FURTHER MATHS SS 2 LESSON NOTE

Week 1

Binomial Distribution

In probability theory and statistics, the binomial distribution is the discrete probability distribution that gives only two possible results in an experiment, either Success or Failure. For example, if we toss a coin, there could be only two possible outcomes: heads or tails, and if any test is taken, then there could be only two results: pass or fail. This distribution is also called a binomial probability distribution.

There are two parameters n and p used here in a binomial distribution. The variable 'n' states the number of times the experiment runs and the variable 'p' tells the probability of any one outcome. Suppose a die is thrown randomly 10 times, then the probability of getting 2 for anyone throw is ⅙. When you throw the dice 10 times, you have a binomial distribution of n = 10 and p = ⅙. Learn the formula to calculate the two outcome distribution among multiple experiments along with solved examples here in this article.

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Binomial Probability Distribution

In binomial probability distribution, the number of 'Success' in a sequence of n experiments, where each time a question is asked for yes-no, then the boolean-valued outcome is represented either with success/yes/true/one (probability p) or failure/no/false/zero (probability q = 1 − p). A single success/failure test is also called a Bernoulli trial or Bernoulli experiment, and a series of outcomes is called a Bernoulli process. For n = 1, i.e. a single experiment, the binomial distribution is a Bernoulli distribution. The binomial distribution is the base for the famous binomial test of statistical importance.

Negative Binomial Distribution

In probability theory and statistics, the number of successes in a series of independent and identically distributed Bernoulli trials before a particularised number of failures happens. It is termed as the negative binomial distribution. Here the number of failures is denoted by ‘r’. For instance, if we throw a dice and determine the occurrence of 1 as a failure and all non-1’s as successes. Now, if we throw a dice frequently until 1 appears the third time, i.e., r = three failures, then the probability distribution of the number of non-1s that arrived would be the negative binomial distribution.

Binomial Distribution Examples

As we already know, binomial distribution gives the possibility of a different set of outcomes. In real life, the concept is used for:

Finding the quantity of raw and used materials while making a product.

Taking a survey of positive and negative reviews from the public for any specific product or place.

By using the YES/ NO survey, we can check whether the number of persons views the particular channel.

To find the number of male and female employees in an organisation.

The number of votes collected by a candidate in an election is counted based on 0 or 1 probability.

Also, read:

Standard Normal Distribution

Cumulative Frequency Distribution

Frequency Distribution Table

Probability Distribution

Probability Class 11

Probability For Class 12

Binomial Distribution Formula

The binomial distribution formula is for any random variable X, given by;

$P(x:n,p) = nCx\ px\ (1-p)n-x$
Or
$P(x:n,p) = nCx\ px\ (q)n-x$
Where,
n = the number of experiments
x = 0, 1, 2, 3, 4, …
p = Probability of Success in a single experiment
q = Probability of Failure in a single experiment = 1 – p
The binomial distribution formula can also be written in the form of n-Bernoulli trials, where nCx = n!/x!(n-x)!. Hence,
$P(x:n,p) = n!/[x!(n-x)!].px.(q)n-x$

Binomial Distribution Mean and Variance
For a binomial distribution, the mean, variance and standard deviation for the given number of success are represented using the formulas
Mean, $\mu = np$
Variance, $\sigma 2 = npq$
Standard Deviation $\sigma = \surd(npq)$
Where p is the probability of success
q is the probability of failure, where q = 1-p

Binomial Distribution Vs Normal Distribution
The main difference between the binomial distribution and the normal distribution is that binomial distribution is discrete, whereas the normal distribution is continuous. It means that the binomial distribution has a finite amount of events, whereas the normal distribution has an infinite number of events. In case, if the sample size for the binomial distribution is very large, then the distribution curve for the binomial distribution is similar to the normal distribution curve.

Properties of Binomial Distribution
The properties of the binomial distribution are:
There are two possible outcomes: true or false, success or failure, yes or no.
There is ‘n’ number of independent trials or a fixed number of n times repeated trials.

The probability of success or failure varies for each trial.
Only the number of success is calculated out of n independent trials.
Every trial is an independent trial, which means the outcome of one trial does not affect the outcome of another trial.
Binomial Distribution Examples And Solutions

Example 1:

If a coin is tossed 5 times, find the probability of:
(a) Exactly 2 heads
(b) At least 4 heads.
Solution:
(a) The repeated tossing of the coin is an example of a Bernoulli trial.
According to the problem:
Number of trials: $n=5$
Probability of head: $p= 1/2$ and hence the probability of tail, $q =1/2$
For exactly two heads:
$x=2$
$P(x=2) = {}^5C_2\, p^2\, q^{5-2} = 5! / 2!\, 3! \times (½)^2 \times (½)^3$
$P(x=2) = 5/16$
(b) For at least four heads,
$x \geq 4$, $P(x \geq 4) = P(x = 4) + P(x=5)$
Hence,
$P(x = 4) = {}^5C_4\, p^4\, q^{5-4} = 5!/4!\, 1! \times (½)^4 \times (½)^1 = 5/32$
$P(x = 5) = {}^5C_5\, p^5\, q^{5-5} = (½)^5 = 1/32$
Therefore,
$P(x \geq 4) = 5/32 + 1/32 = 6/32 = 3/16$

Example 2:

For the same question given above, find the probability of:
a) Getting at least 2 heads

Solution: P (at most 2 heads) = $P(X \leq 2) = P(X = 0) + P(X = 1)$
$P(X = 0) = (½)5 = 1/32$
$P(X=1) = 5C1\ (½)5. = 5/32$
Therefore,
$P(X \leq 2) = 1/32 + 5/32 = 3/16$

Example 3:

A fair coin is tossed 10 times, what are the probability of getting exactly 6 heads and at least six heads.
Solution:
Let x denote the number of heads in an experiment.
Here, the number of times the coin tossed is 10. Hence, n=10.
The probability of getting head, p ½
The probability of getting a tail, q = 1-p = 1-(½) = ½.
The binomial distribution is given by the formula:
P(X= x) = nCxpxqn-x, where = 0, 1, 2, 3, …
Therefore, P(X = x) = 10Cx(½)x(½)10-x
(i) The probability of getting exactly 6 heads is:
P(X=6) = 10C6(½)6(½)10-6
P(X= 6) = 10C6(½)10
P(X = 6) = 105/512.
Hence, the probability of getting exactly 6 heads is 105/512.
(ii) The probability of getting at least 6 heads is $P(X \geq 6)$
$P(X \geq 6) = P(X=6) + P(X=7) + P(X= 8) + P(X = 9) + P(X=10)$
$P(X \geq 6)$ = 10C6(½)10 + 10C7(½)10 + 10C8(½)10 + 10C9(½)10 + 10C10(½)10
$P(X \geq 6) = 193/512$.

Practice Problems

Solve the following problems based on binomial distribution:
The mean and variance of the binomial variate X are 8 and 4 respectively. Find $P(X<3)$.

The binomial variate X lies within the range {0, 1, 2, 3, 4, 5, 6}, provided that P(X=2) = 4P(x=4). Find the parameter "p" of the binomial variate X.
In binomial distribution, X is a binomial variate with n= 100, p= ⅓, and P(x=r) is maximum. Find the value of r.
Probability is a wide and very important topic for class 11 and class 12 students. By capturing the concepts here at BYJU'S, students can excel in the exams.

Frequently Asked Questions on Binomial Distribution

What is meant by binomial distribution?

The binomial distribution is the discrete probability distribution that gives only two possible results in an experiment, either success or failure.

Mention the formula for the binomial distribution.

The formula for binomial distribution is:
P(x: n,p) = nCx px (q)n-x
Where p is the probability of success, q is the probability of failure, n= number of trials

What is the formula for the mean and variance of the binomial distribution?
The mean and variance of the binomial distribution are:

Mean = np
Variance = npq

What are the criteria for the binomial distribution?
The number of trials should be fixed.
Each trial should be independent.
The probability of success is exactly the same from one trial to the other trial.

What is the difference between a binomial distribution and normal distribution?

The binomial distribution is discrete, whereas the normal distribution is continuous.

Week 2

Poisson Probability distribution Examples and Questions

Poisson probability distribution is used in situations where events occur randomly and independently a number of times on average during an interval of time or space. The random variable XX associated with a Poisson process is discrete and therefore the Poisson distribution is discrete.

Poisson Process Examples and Formula

Example 1

These are examples of events that may be described as Poisson processes:

My computer crashes on average once every 4 months.

Hospital emergencies receive on average 5 very serious cases every 24 hours.

The number of cars passing through a point, on a small road, is on average 4 cars every 30 minutes.

I receive on average 10 e-mails every 2 hours.

Customers make on average 10 calls every hour to the customer help center

Conditions for a Poisson distribution are

1) Events are discrete, random and independent of each other.

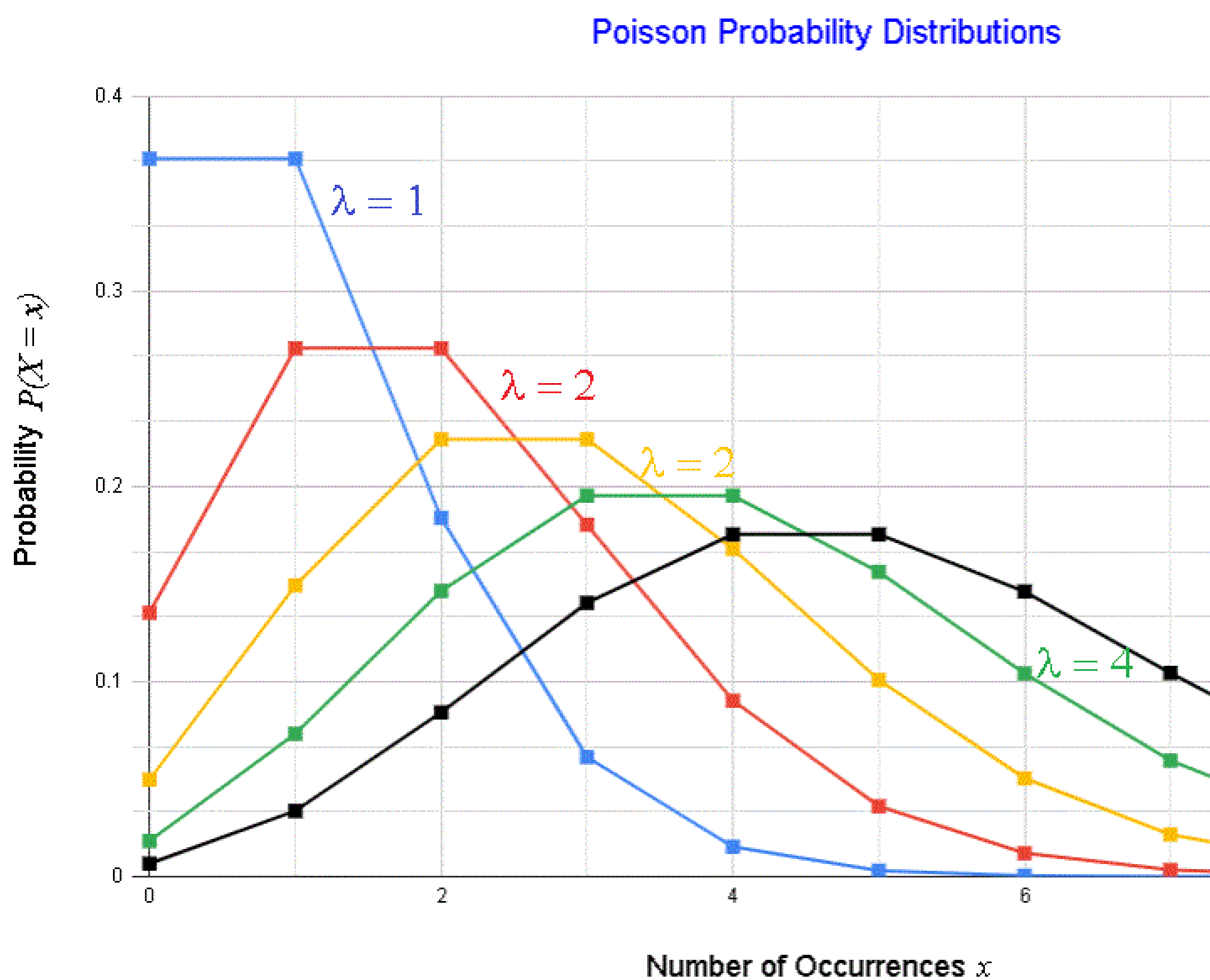
2) The average number of times of occurrence of the event is constant over the same period of time.

3) Probabilities of occurrence of event over fixed intervals of time are equal.

4) Two events cannot occur at the same time; they are mutually exclusive.

In a Poisson distribution, if an event happens an average λλ times over a period TT of time or space, the probability that it will happen xx times over a period of time TT is given by

P(X=x)=e−λλxx!P(X=x)=e−λλxx!
where e≈2.7182818e≈2.7182818 is the base of the natural logarithm, x!x! is the factorial of xx defined as x!=1×2×3...×(x−1)×xx!=1×2×3...×(x−1)×x and x=0,1,2,....∞x=0,1,2,....∞
Below we show the graphs of P(X)P(X) for several values of the average λλ and we note that the probability is maximum for xx close to the average λλ and decreases as xx takes larger values which makes sense.



Compare Binomial and Poisson Distributions
A binomial distribution has two parameters: the number of trials nn and the probability of success pp at each trial while a Poisson distribution has one parameter which is the average number of times λλ that the event occur over a fixed period of time.
In the binomial distribution xx is an integer taking values over the interval [0,n][0,n] , while in the Poisson distribution xx is an integer taking values over the interval [0,∞)[0,∞)

Week 3

Normal Distribution | Examples, Formulas, & Uses

In a normal distribution, data is symmetrically distributed with no skew. When plotted on a graph, the data follows a bell shape, with most values clustering around a central region and tapering off as they go further away from the center.
Normal distributions are also called Gaussian distributions or bell curves because of their shape.

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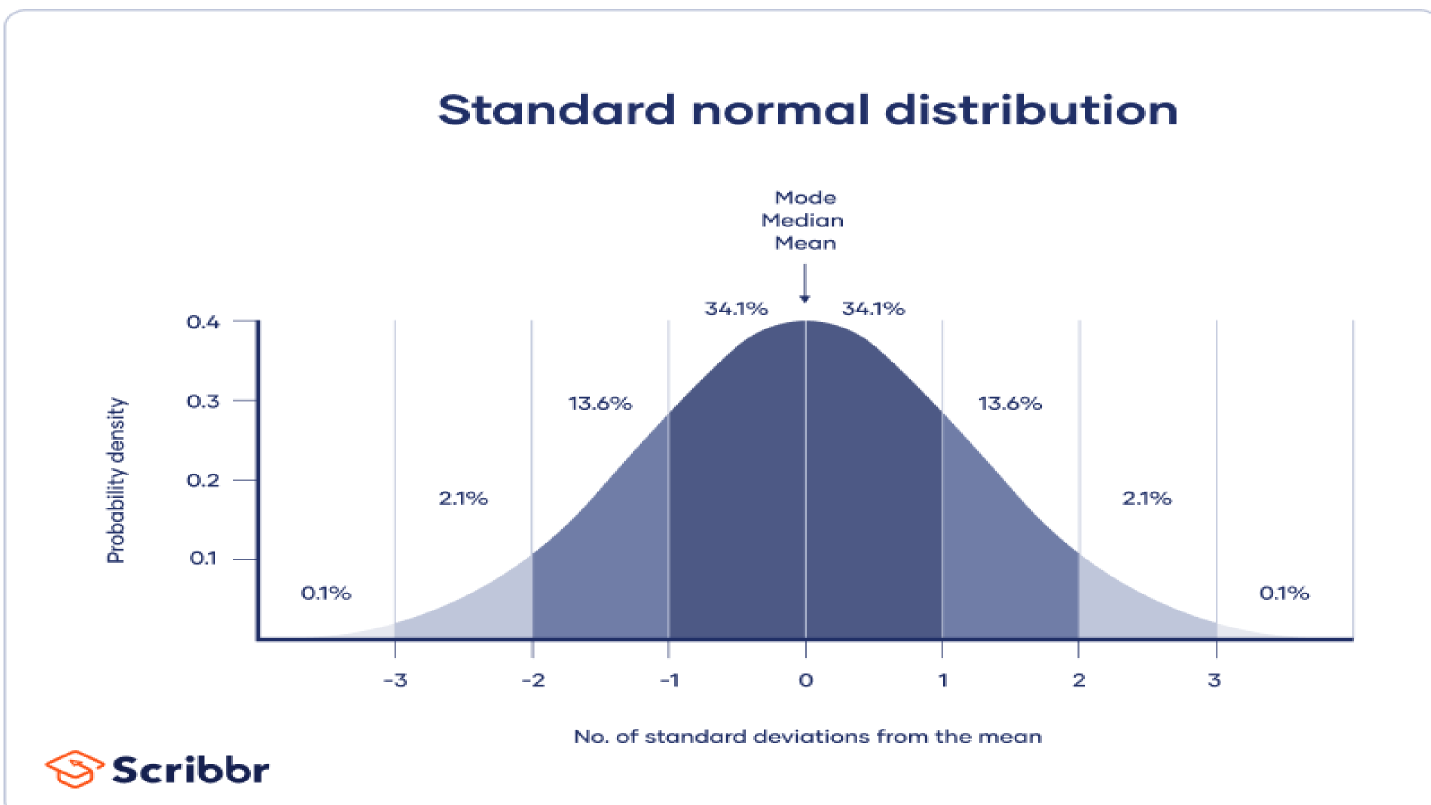
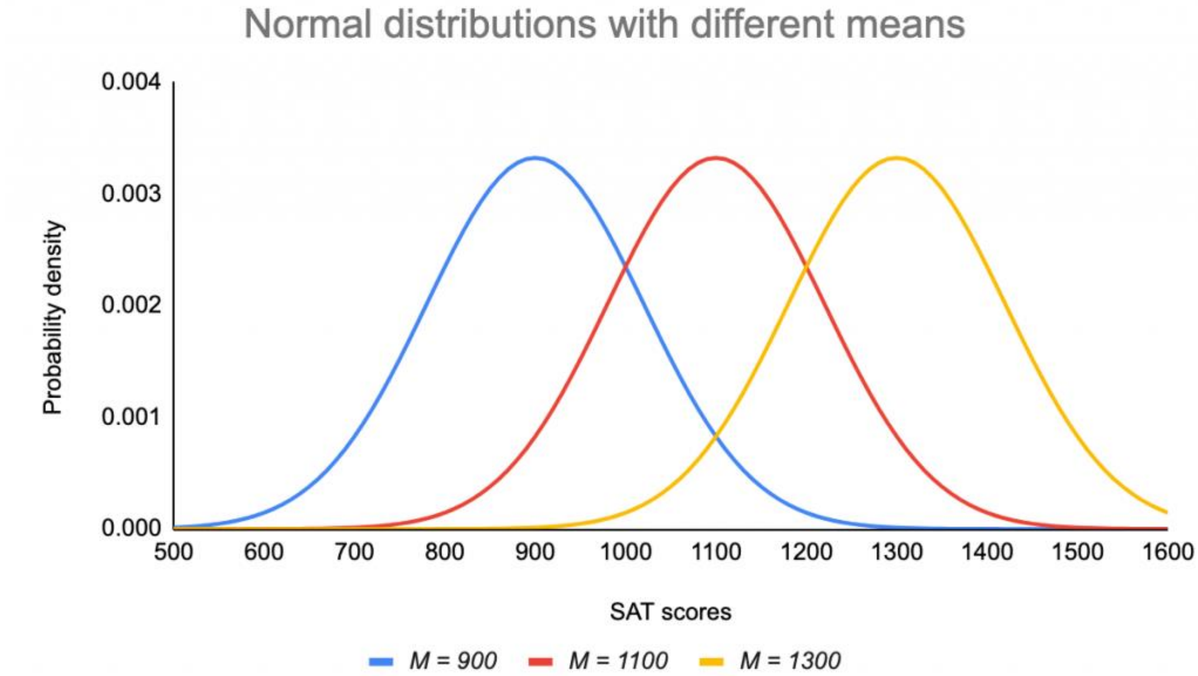
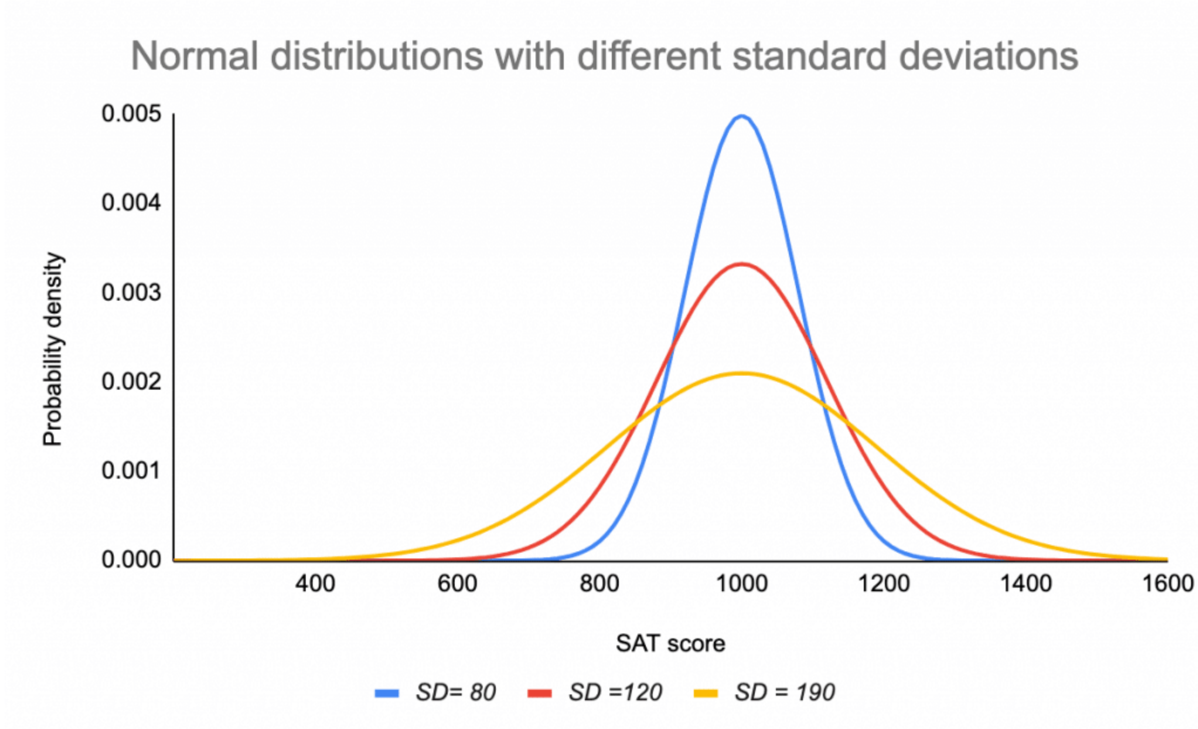
Frequently asked questions about normal distributions
Why do normal distributions matter?
All kinds of variables in natural and social sciences are normally or approximately normally distributed. Height, birth weight, reading ability, job satisfaction, or SAT scores are just a few examples of such variables.
Because normally distributed variables are so common, many statistical tests are designed for normally distributed populations.
Understanding the properties of normal distributions means you can use inferential statistics to compare different groups and make estimates about populations using samples.
What are the properties of normal distributions?
Normal distributions have key characteristics that are easy to spot in graphs:
The mean, median and mode are exactly the same.
The distribution is symmetric about the mean—half the values fall below the mean and half above the mean.
The distribution can be described by two values: the mean and the standard deviation.



The mean is the location parameter while the standard deviation is the scale parameter.
The mean determines where the peak of the curve is centered. Increasing the mean moves the curve right, while decreasing it moves the curve left.



The standard deviation stretches or squeezes the curve. A small standard deviation results in a narrow curve, while a large standard deviation leads to a wide curve.



Normal distributions with different standard deviations" width="600" height="371"

Empirical rule

The empirical rule, or the 68-95-99.7 rule, tells you where most of your values lie in a normal distribution:

Around 68% of values are within 1 standard deviation from the mean.

Around 95% of values are within 2 standard deviations from the mean.

Around 99.7% of values are within 3 standard deviations from the mean.

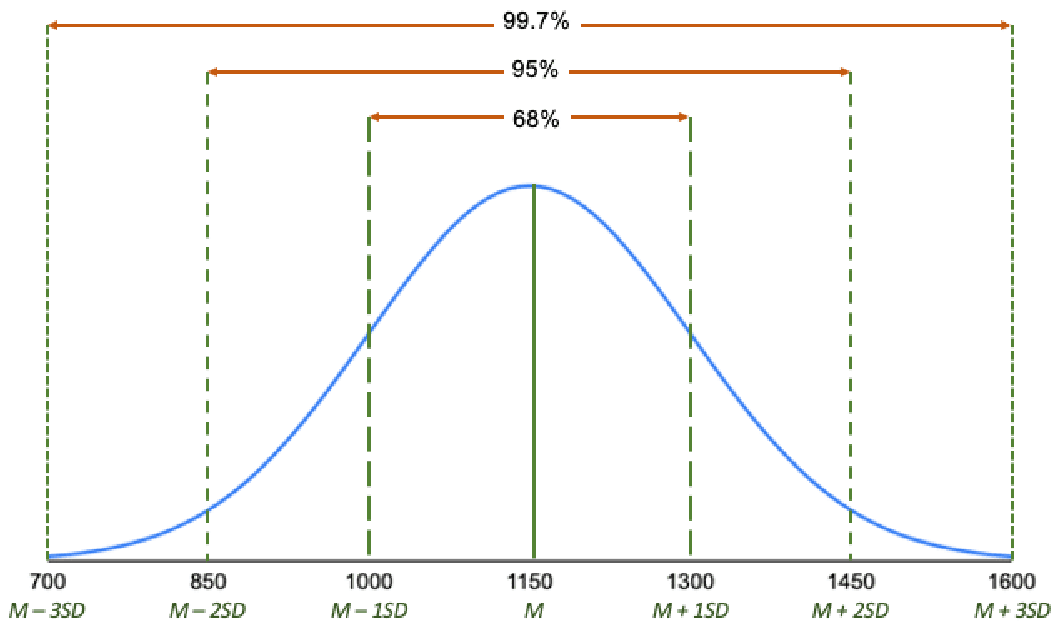
Example: Using the empirical rule in a normal distribution You collect SAT scores from students in a new test preparation course. The data follows a normal distribution with a mean score (M) of 1150 and a standard deviation (SD) of 150.

Following the empirical rule:

Around 68% of scores are between 1000 and 1300, 1 standard deviation above and below the mean.

Around 95% of scores are between 850 and 1450, 2 standard deviations above and below the mean.

Around 99.7% of scores are between 700 and 1600, 3 standard deviations above and below the mean.



The empirical rule is a quick way to get an overview of your data and check for any outliers or extreme values that don't follow this pattern.

If data from small samples do not closely follow this pattern, then other distributions like the t-distribution may be more appropriate. Once you identify the distribution of your variable, you can apply appropriate statistical tests.

Central limit theorem

The central limit theorem is the basis for how normal distributions work in statistics.

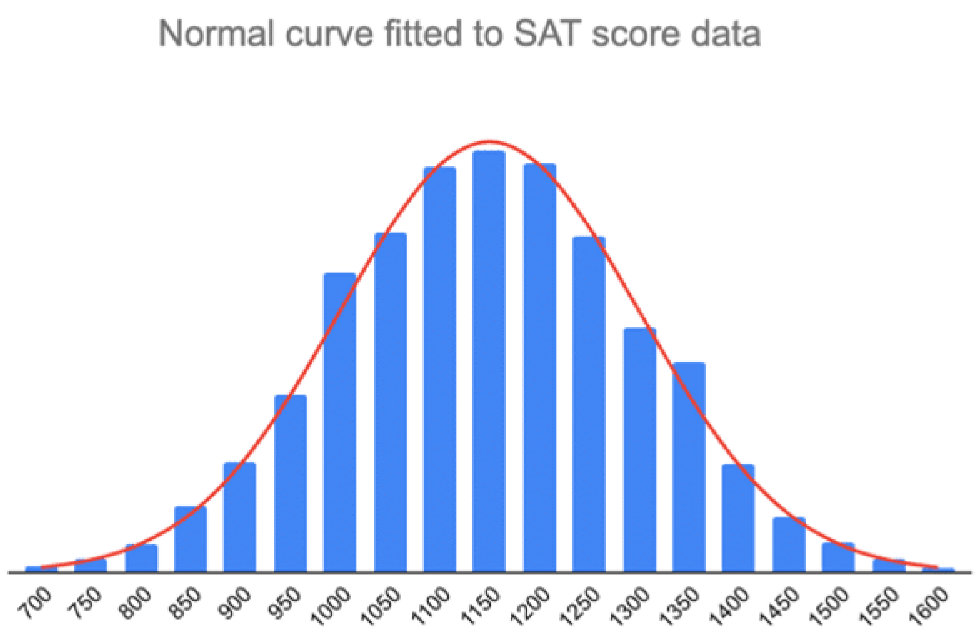
In research, to get a good idea of a population mean, ideally you'd collect data from multiple random samples within the population. A sampling distribution of the mean is the distribution of the means of these different samples.

The central limit theorem shows the following:

Law of Large Numbers: As you increase sample size (or the number of samples), then the sample mean will approach the population mean.

With multiple large samples, the sampling distribution of the mean is normally distributed, even if your original variable is not normally distributed.

Parametric statistical tests typically assume that samples come from normally distributed populations, but the central limit theorem means that this assumption isn't necessary to meet when you have a large enough sample. You can use parametric tests for large samples from populations with any kind of distribution as long as other important assumptions are met. A sample size of 30 or more is generally considered large.
For small samples, the assumption of normality is important because the sampling distribution of the mean isn't known. For accurate results, you have to be sure that the population is normally distributed before you can use parametric tests with small samples.
Formula of the normal curve
Once you have the mean and standard deviation of a normal distribution, you can fit a normal curve to your data using a probability density function.



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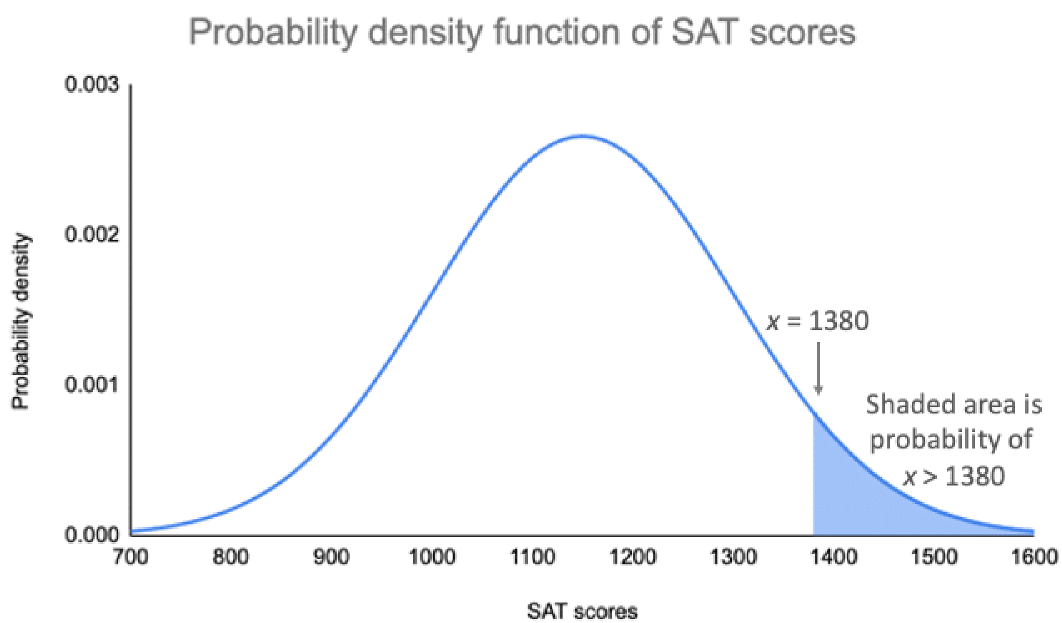
In a probability density function, the area under the curve tells you probability. The normal distribution is a probability distribution, so the total area under the curve is always 1 or 100%.
The formula for the normal probability density function looks fairly complicated. But to use it, you only need to know the population mean and standard deviation.

For any value of x, you can plug in the mean and standard deviation into the formula to find the probability density of the variable taking on that value of x.

Normal Probability Density Formula	Explanation
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$	f(x) = probability x = value of the variable μ = mean σ = standard deviation σ2 = variance

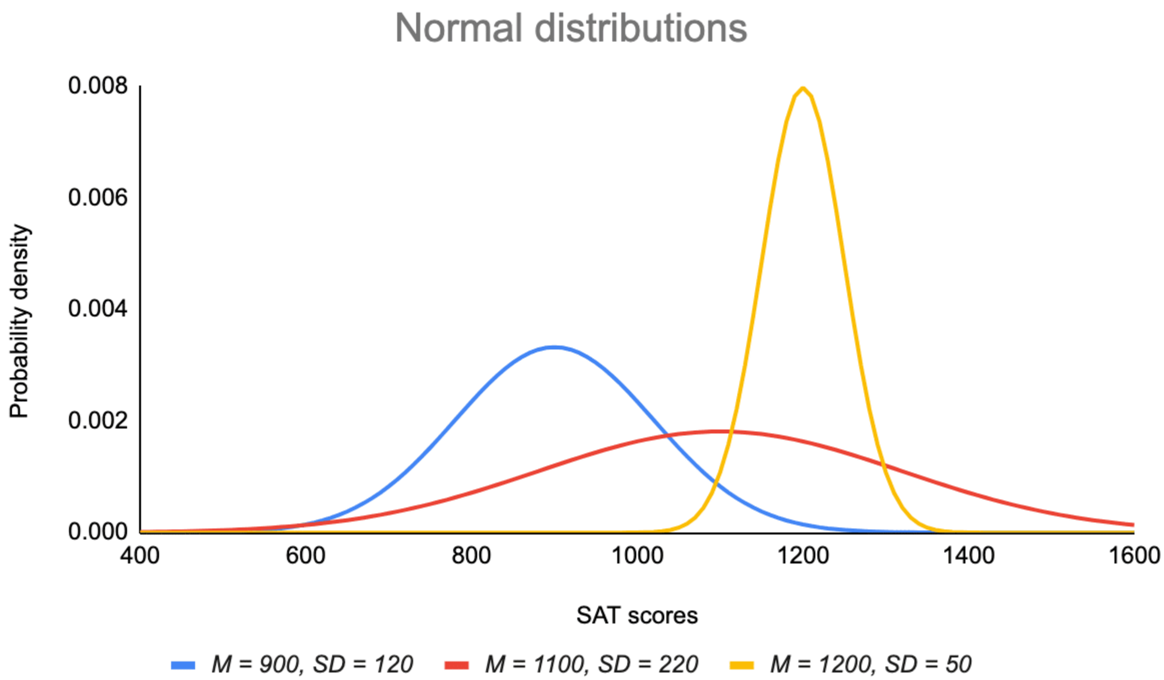
Example: Using the probability density functionYou want to know the probability that SAT scores in your sample exceed 1380.

On your graph of the probability density function, the probability is the shaded area under the curve that lies to the right of where your SAT scores equal 1380.

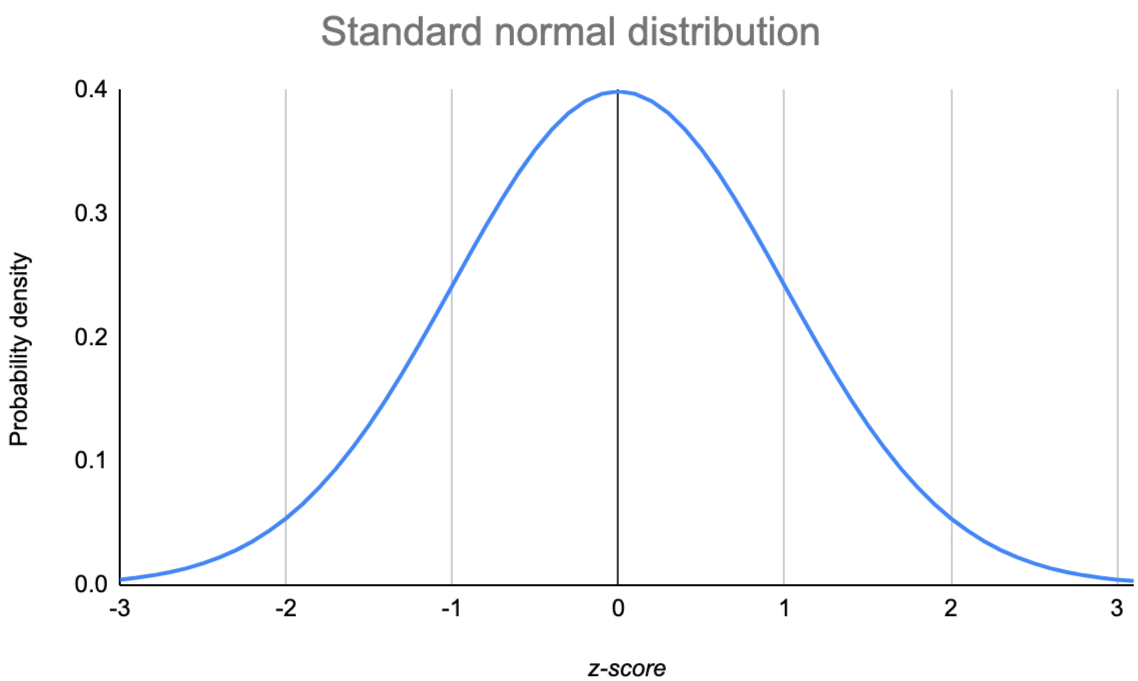


You can find the probability value of this score using the standard normal distribution.

What is the standard normal distribution?
The standard normal distribution, also called the z-distribution, is a special normal distribution where the mean is 0 and the standard deviation is 1. Every normal distribution is a version of the standard normal distribution that's been stretched or squeezed and moved horizontally right or left.



While individual observations from normal distributions are referred to as x, they are referred to as z in the z-distribution. Every normal distribution can be converted to the standard normal distribution by turning the individual values into z-scores.
Z-scores tell you how many standard deviations away from the mean each value lies.



You only need to know the mean and standard deviation of your distribution to find the z-score of a value.

Z-score Formula	Explanation
$$z = \frac{x - \mu}{\sigma}$$	x = individual value μ = mean σ = standard deviation

We convert normal distributions into the standard normal distribution for several reasons:
To find the probability of observations in a distribution falling above or below a given value.
To find the probability that a sample mean significantly differs from a known population mean.
To compare scores on different distributions with different means and standard deviations.
Finding probability using the z-distribution

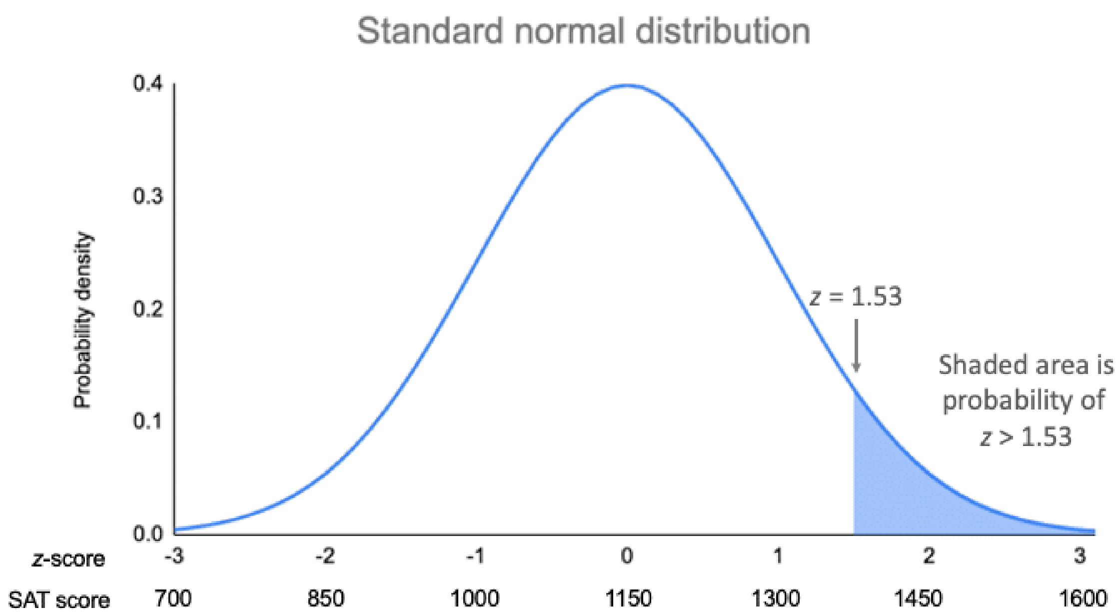
Each z-score is associated with a probability, or p-value, that tells you the likelihood of values below that z-score occurring. If you convert an individual value into a z-score, you can then find the probability of all values up to that value occurring in a normal distribution.

Example: Finding probability using the z-distributionTo find the probability of SAT scores in your sample exceeding 1380, you first find the z-score.

The mean of our distribution is 1150, and the standard deviation is 150. The z-score tells you how many standard deviations away 1380 is from the mean.

Formula	Calculation
$z = (x - \mu) / \sigma$	$z = (1380 - 1150) / 150$ $z = 1.53$

For a z-score of 1.53, the p-value is 0.937. This is the probability of SAT scores being 1380 or less (93.7%), and it's the area under the curve left of the shaded area.



To find the shaded area, you take away 0.937 from 1, which is the total area under the curve.

Probability of $x>1380 = 1 - 0.937 = 0.063$

That means it is likely that only 6.3% of SAT scores in your sample exceed 1380.

Frequently asked questions about normal distributions
What is a normal distribution?
In a normal distribution, data is symmetrically distributed with no skew. Most values cluster around a central region, with values tapering off as they go further away from the center.
The measures of central tendency (mean, mode and median) are exactly the same in a normal distribution.
What is a standard normal distribution?
The standard normal distribution, also called the z-distribution, is a special normal distribution where the mean is 0 and the standard deviation is 1.
Any normal distribution can be converted into the standard normal distribution by turning the individual values into z-scores. In a z-distribution, z-scores tell you how many standard deviations away from the mean each value lies.
What is the empirical rule?
The empirical rule, or the 68-95-99.7 rule, tells you where most of the values lie in a normal distribution:
Around 68% of values are within 1 standard deviation of the mean.
Around 95% of values are within 2 standard deviations of the mean.
Around 99.7% of values are within 3 standard deviations of the mean.
The empirical rule is a quick way to get an overview of your data and check for any outliers or extreme values that don’t follow this pattern.
What is a t-distribution?
The t-distribution is a way of describing a set of observations where most observations fall close to the mean, and the rest of the observations make up the tails on either side. It is a type of normal distribution used for smaller sample sizes, where the variance in the data is unknown.
The t-distribution forms a bell curve when plotted on a graph. It can be described mathematically using the mean and the standard deviation.

T-distribution: What it is and how to use it

The t-distribution is a type of normal distribution that is used with small sample sizes, where the variance of a sample is unknown.
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The standard normal distribution
In the standard normal distribution, the mean is 0 and the standard deviation is 1. A normal distribution can be standardized using z-scores.
328

The p-value explained
The p-value shows the likelihood of your data occurring under the null hypothesis. P-values help determine statistical significance.
500

Week 4 and 5

Equilibrium of Forces Acting at a Point
Pre-lab Questions 1. What is the definition of equilibrium? Can an object be moving and still be in equilibrium? Explain. 2. For this lab, what criteria will you use to decide when the forces acting on the ring are in equilibrium? 3. Draw a set of three vectors whose sum is zero. 4. How will you estimate the uncertainty of each force acting on the ring? 5. How will you estimate the uncertainty of each angle measurement? 6. How much error will be introduced if the table is not level? How could you test this empirically? 7. What are the units of sine and cosine? Does your answer depend on the units of the angle (degrees or radians)?
Introduction

Addition of Forces
Forces are one of a group of quantities known as vectors, which are distinguished from regular number (know as scalars) by the fact that a vector has two quantities associated with it, a magnitude and a direction (related to a coordinate axes of the system you are dealing). These properties completely characterize a vector.
A vector may alternatively be described by specifying its vector components. In the case of the Cartesian coordinate system (the system we will be primarily dealing with) there are two components, the x-component and y-component. These two properties also completely characterize a vector.
Vectors, and in the case of this lab, force vectors, can be represented pictorially (see Fig.1) by an arrow pointing in the direction of action of the force, with a length proportional to the strength (magnitude) of the force.
The components Fx and Fy in the x and y directions of the vector F are related to the magnitude F and angle θ by:
θ cosFF x = and
θ
sinFF y =
and conversely:
2 2 , and =arctan y x y x F F F F F θ ⍰ ⍰ = + ⍰ ⍰ ⍰ ⍰
.
F
Fy
θ
Fx
Figure 1
Equilibrium of Forces Acting at a Point
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When several forces act on a point, their sum can be obtained according to the rules of vector algebra. Graphically, the sum of two forces F = F1 + F2 can be found by using the parallelogram rule illustrated in Fig. 2a or equivalently, by the head-to-tail method illustrated in Fig. 2b.
The sum of the vectors can also be derived analytically by adding their components: Fx = F1x+F2x, and Fy = F1y+F2y .
Condition for Translational Equilibrium

An object is in translational equilibrium when the vector sum of all the forces acting on it is zero. In this experiment we shall study the translational equilibrium of a small ring acted on by several forces on an apparatus known as a force table, see Fig. 3. This apparatus enables one to cause the forces of gravity acting on several masses (F = mg) to be brought to bear on the small ring. These forces are adjusted until equilibrium of the ring is achieved. You will then add the forces analytically by adding their components and graphically by drawing the vectors and determining if they add to zero using the rules for the addition of force vectors listed above.

Procedure

Part 1. Equilibrium with Three Forces

We shall first study the equilibrium of the small ring when there are three forces acting on it. Two of the forces (F1 and F2) will be fixed and the third one F3 adjusted until equilibrium is reached.

1. If necessary, level the force table using the small bubble level placed on the table's surface.

2. Choose any two masses you like in the range 100-300 g, and place each mass on a weight holder. Designate the measured masses as m1 and m2, and use an electronic balance to measure each of the masses including the holder. The uncertainty of these measurements will be determined by either the precision or accuracy of the balance, whichever is greater (see Instrument Accuracy Ratings)

2
F
F
F F
F
F
2
1 1 Parallelogram Rule Head-to-Tail Method
Figure 2a Figure 2b

3. Place the pin in the middle of the force table and place the ring over the pin. Attach two of the four pulleys provided to the force table at any position

other than zero degrees. Record the value of 1 θ and 2 θ . The uncertainty in these angles should be limited to the precision to which you can read the angles on the force table.

4. Run two of the strings (attached to the ring) over the pulleys, and suspend the masses that you have chosen at the appropriate angles (m1 at 1 θ and m2 at 2 θ). The two strings pull on the ring with tension forces F1 and F2, each with a magnitude approximately equal to the weight of the corresponding mass and holder (m1g, and m2g) suspended at the end of each of the string.

5. Pull one of the remaining strings in various directions until you locate an angle in which the ring is freed from the pin when you apply the right amount of force. Attach a third pulley at this position. Run the string over the pulley and attach a weight holder to the string. Add weights to the weight holder until the ring is centered around the pin and pulls away from the pin, so that the pin is not necessary to hold the ring in place. This last added force is the (equilibriant) force F3 (m3g). It may be necessary to make minor adjustments to the angle to obtain a precise measurement. Make sure that the strings are stretched radially and the pin is at the center of the ring. Gently tapping on the table will reduce the effect of friction from the pulleys. Estimate the uncertainty in the equilibrant force by adjusting the mass and angle until the system is no longer in equilibrium.

Part 2. Equilibrium with Four Forces

1. Now select three masses (to provide three forces with sum F1 + F2 + F3) at three angles (one of them zero) and determine what fourth single mass and angle establishes equilibrium on the force table (the equilibriant force F4).

2. Record all angles, masses and their uncertainties as in part 1.

Be sure to pledge your work, initial your data, and have your TA initial your data.

Analysis

Part 1. Graphical Analysis

Make accurate diagrams on rectangular graph paper showing the sum of the forces acting on the ring for both parts of the experiment above (equilibrium for 3 and 4 forces)

1. Draw force diagrams to scale. For example, 5 Newtons = 1 cm. Use whatever scale works best to give you the greatest plotting precision.

2. Use the head-to-tail method to find the sum of the forces graphically. Be as accurate as possible. Qualitatively verify that the sum is zero. If it is not, determine from your graph the magnitude of the deviation from zero.
Equilibrium of Forces Acting at a Point
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Part 2. Analytical Sum

Calculate the resultant force on the ring, FT = F1 + F2 + F3, analytically for Part 1 only. Choose zero degree to be the +x-axis, and 90° to be the +y-axis. A data analysis sheet is provided to facilitate the error analysis.

1. Use the tables on the data analysis sheet to enter the data for the forces acting on the ring. For each force, include the magnitude F, its uncertainty uF, the direction θ , and its uncertainty u θ . The values of u θ must be expressed in units of radians.
2. Calculate the x- and the y-components of each of the forces together with their errors. Pay attention to the sign of each component. Include in the last row of the table the sum of the components and their error.
3. Calculate the magnitude and the direction of the resultant force FT. Compute also the uncertainty of the magnitude.

Discussion Is the condition for static equilibrium, FT = 0, satisfied for both parts of the experiment? How does your uncertainty of FT compare to the precision of your force and angle measurements? Discuss the sources of systematic error and how they affect your results. What is the primary source of error in this experiment? Discuss attempts you have made to reduce both systematic and random errors. What did you learn or discover from this lab? When might you apply the skills learned from this lab?

Equilibrium of Forces Acting at a Point
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DATA ANALYSIS

Magnitude and Direction of Applied Forces:
m (g) um (g) F (N) uF (N)
θ (deg) u θ (deg) Force 1
Force 2
Force 3
Sample Calculation: (Force 3) Note: Newton (N) = kg*m/s2

F = mg =

____

uF = gum =

____

Components of the Applied Forces:

Fx (N) uFx(N) Fy (N) uFy (N)

Force 1

Force 2

Force 3

Resultant

Sample Calculation: (Components of Force 3)

F F cos x θ =

=____

xFu

=____
_

F F sin y θ =

=____

yFu

=____
_

Equilibrium of Forces Acting at a Point

Week 6, 7 and 8

Mathematical Modeling

Mathematical modeling is becoming an increasingly important subject as computers expand our ability to translate mathematical equations and formulations into concrete conclusions concerning the world, both natural and artificial, that we live in.

1.1 EXAMPLES OF MODELING

Here we do a quick tour of several examples of the mathematical process. We present the models as finished results as opposed to attempting to develop the models.

1.1.1 Modeling with Difference Equations

Consider the situation in which a variable changes in discrete time steps. If the current value of the variable is a_n then the predicted value of the variable will be a_{n+1}. A mathematical model for the evolution of the (still unspecified) quantity a_n could take the form $a_{n+1} = \alpha a_n + \beta$ In words, the new value is a scalar multiple of the old value offset by some constant β. This model is common, e.g., it is used for modeling bank loans. One might amend the model to make the dependence depend on more terms and to include the possibility that every iteration the offset can change, thus, $a_{n+1} = \alpha_1 a_n + \alpha_2 a_n^2 + \beta_n$

This could correpsond to, for example, a population model where the the migration levels change every time step. In some instances, it is clear that information required to predict a new value goes back further than the current value, e.g.,

$a_{n+1} = a_n + a_{n-1}$

Note now that two initial values are required to evolve this model. Finally, it may be that the form of the difference equations are unknown and the model must be written $a_{n+1} = f(a_n, a_{n-1}, a_{n-M-1})$ Determining the nature of f and the step M is at the heart of model formulation with difference equations. Often observed data can be employed to assist in this effort.

1.1.2 Modeling with Ordinary Differential Equations

Although modeling with ordinary differential equations shares many of the ideas of modeling with the difference equations discussed above, there are many fundamen

tal differences. At the center of these differences is the assumption that time is a continuous variable. One of the simplest differential equations is also an extremely important model, i.e., $\frac{dx}{dt} = \alpha x$ In words, the rate of change of the quantity x depends on the amount of the quantity. If $\alpha > 0$ then we have exponential growth. If $\alpha < 0$ the situation is exponetial decay. Of course additional terms can be added that fundamentally alter the evolution of $x(t)$. For example

$$\frac{dx}{dt} = \alpha_1 x + \alpha_2 x^2$$

The model formulation again requires the development of the appropriate righthand side. In the above model the value x on the right hand side is implicitly assumed to be evaluated at the time t. It may be that there is evidence that the instantaneous rate of change at time t is actually a function of a previous time, i.e.,

$$\frac{dx}{dt} = f(x(t)) + g(x(t-\tau))$$

This is referred to as a delay differential equation.

1.1.3 Modeling with Partial Differential Equation

In the previous sections on modeling the behaviour of a variable as a function of time we assumed that there was only one independent variable. Many situations arise in practice where the number of independent variables is larger than two. For spatio-temporal models we might have time and space (hence the name!), e.g.,

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

or

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

1.1.4 Optimization

In many modeling problems the goal is to compute the ”best” solution. This may correspond to maximizing profit in a company, or minimizing loss in a conflict. It is no surprise that optimization techniques take a central seat in the mathematical modeling literature. Now one may allow $x \in R^n$ and require that $x^* = \operatorname{argmin} f(x)$

The quantity $f(x)$ is referred to as the objective function while the vector x consists of decision variables. Because x sits in R^n the problem is referred to as unconstrained.

Alternatively, one might require that the solution x have all positive components. If we refer to this set as S then the optimization problem is constrained

$x* = \text{argmin}_{x \in S}$

$f(x)$

If the objective function as well as the equations that define the constraint set are linear, than the optimization problem is called a linear programming problem. Otherwise, the problem is referred to as a nonlinear programming problem. As we shall see, solution methods for linear and nonlinear programming problems are very different.

1.1.5 Modeling with Simulations

Many problems may afford a mathematical formulation yet be analytically intractable. In these situations a computer can implement the mathematics literally and repetitively often times to extreme advantage.

Simulating Games. • What is the probability that you can win a game of solitaire? • What is the best strategy for playing blackjack? • Given a baseball team consisting of certain players, in what order should they hit?

On the other hand, computer simulations can be employed to model evolution equations. Applications in the realm of fluid dynamics and weather prediction are well established. A striking new example of such simulation modeling is attempting to model electrical activity in the brain.

1.1.6 Function Fitting: Data Modeling

Often data is available from a process to assist in the modeling. How can functions be computed that reflect the relationships between variables in the data. Produce a model $y = f(x;w)$

and using the set of input output pairs compute the parameters w. In some cases the form of f may be guessed. In other cases a model free approach can be used.

1.2 THE MODELING PROCESS

The goal in all modeling problems is added value. Something novel must be learned from the modeling process or one has completed an exercise in futility, or mathematical wheel spinning, depending on your perspective. There are many obvious questions the answers to which have inherent added value. For example: • Should a stock be bought or sold? • Is the earth becoming warmer?

• Does creating a law have a positive or negative societal effect? • What is the most valuable property in monopoly? Clearly this is a very small start to an extremely long list.

1.2.1 An Algorithm for Modeling?

The modeling process has a sequence of common steps that serve as an abstraction for the modeler: • Identify the problem and questions. • Identify the relevant variables in a problem. • Simplify until tractable. • Relate these variables mathematically. • Solve. • Does the solution provide added value? • Tweak model and compare solutions.

1.3 THE DELICATE SCIENCE OF ERRORS

If one had either infinite time or infinite computing power error analysis would presumably be a derelict activity: all models would be absolutely accurate. Obviously, in reality, this is not the case and a well-accepted modus operandi in modeling is committing admissible errors. Of course, in practice, the science is more ad hoc. If terms in an equation introduce computational difficulties the immediate question arises as to what would happen if those terms are ignored? In theory we would rather keep them but in practice we can't afford to. Thus the delicate science of modeling concerns retaining just enough features to make the model useful but not so many as too make it more expensive to compute than necessary to get out the desirable information. We illustrate this concept by examining the seemingly innocuous junior high school problem $\varrho x^2 + x + 1 = 0$ Of course we can solve this problem exactly using the quadratic formula[1]

$$x = -\frac{1}{2\varrho} \pm \frac{\sqrt{1-4\varrho}}{2\varrho} \tag{1.1}$$

For a moment, let us assume that the quadratic term were actually an unknown term, e.g., $\varrho f(x) + x + 1 = 0$

[1] If you don't recall this, then the famous Science Fiction write Robert Heilein suggested you not be allowed to vote.

and that actually computing f might be rather expensive. We might argue that if ǫ were very small that this term could safely be ignored. Now let us return to the simple case of f(x) = x2. If ǫ is taken as zero then clearly it follows that x = −1 is the unique solution. However, we know from our quadratic equation however that if ǫ = 0.0000001 (any non-zero number would do), then there are two solutions rather than one. So we have actually lost a potentially important solution by ignoring what appeared to be a small quantity. In addition, we may also have introduced inaccuracies into the obtained solution and this issue must be explored. In essence we are concerned with how quickly the solution changes about the point ǫ = 0. A quick graph of Equation (1.1) reveals that the solution changes rather quickly. To see how this solution changes as a function of ǫ consider the series expansion x = a0 + a1ǫ + a2ǫ2 + a3ǫ3 + ...
Substituting this expansion into the original quadratic results in the new equation a0 + 1 + (a2 0 + a1)ǫ + (2a0a1 + a2)ǫ2 +··· = 0 Setting the coefficients of the different powers of ǫ to zero gives the series solution for x as x = −1−ǫ−2ǫ2 + ... (1.2) So if ǫ ≈ 0.01 we can conclude the error is on the order of 1% and the error will grow quickly with ǫ. This problem is explored further in the exercises and function iteration is introduced to track down the 2nd solution in the quadratic equation. For further discussion of these ideas see [4].

1.4 PURPOSE OF THIS COURSE

The primary goal of this course is to assist the student to develop the skills necessary to effectively employ the ideas of mathematics to solve problems. At the simplest level we seek to promote an understanding of why mathematics is useful as a language for characterizing the interaction and relationships among quantifiable concepts, or in mathematical terms, variables. Throughout the text we emphasize the notion of added value and why it is the driving force behind modeling. For a given mathematical model to be deemed a success something must be learned that was not obvious without the modeling procedure. Very often added value comes in the form of a prediction. In the absence of added value the modeling procedure becomes an exercise not unrelated to digging a ditch simply to fill it back up again. The emphasis in this course is on learning why certain mathematical concepts are useful for modeling. We proceed from mathematics to models rather than the popular reverse approach and downplay interdisciplinary expertise required in many specific contexts. We firmly believe that by focusing on mathematical concepts the ability to transfer knowledge from one setting to another will be significantly enhanced. Hence, we emphasize the efficacy of certain mathematics for constructing models.

PROBLEMS 1.1. Name three problems that might be modeled mathematically. Why do you think mathematics may provide a key to each solution. What is the added value in each case? 1.2. Consider the differential equation

$$\frac{dx}{dt} = x$$

Translate this model to a difference equation. Compare the solutions and discuss. 1.3. Consider the equation $x^2 + ϙx - 1 = 0$ for small ϙ. How does ignoring the middle term ϙx change your solution? Is this a serious omission? 1.4. Using a Taylor series expansion express the solution to the quadratic equation in Equation 1.1 as a series. Include terms up to cubic order. 1.5. Find the cubic term in the expansion in Equation (1.2). 1.6. One approach to determining zeros of a general function, i.e., computing roots to $f(x) = 0$, is to rewrite the problem as $f(x) = x - g(x)$ and to employ the iteration $x_{n+1} = g(x_n)$. (a) If we take $g(x) = -\frac{1}{x}$ show that the iteration can be written

$$x_{n+1} = -\frac{1}{ϙ}\left(1 + \frac{1}{x_n}\right)$$

(b) Let $x_0 = -1/ϙ$ and compute x_1. By considering the Taylor series of the solution of the quadratic equation argue that this is a two term approximation to the missing solution. (c) Compute x_2. 1.7. (a) Substitute $x = y/\varepsilon$ into the equation

$$\varepsilon x^2 + x + 1 = 0 \quad (1.3)$$

and multiply the resulting equation for y by ε. Show that this leads to $y^2 + y + \varepsilon = 0$. (1.4) When $\varepsilon = 0$, the equation (1.4) has two solutions: $y = 0$ and $y = -1$. This suggests that (1.4) allows us to compute both solutions of (1.3) through a perturbation analysis. (b) Reproduce the solution to (1.3) given by Eq. (1.2) by computing a solution of the form $y = b_1\varepsilon + b_2\varepsilon^2 + b_3\varepsilon^3 + ...$ for (1.4). (c)

Proceed similarly using the form $y = -1 + c_1\varepsilon + c_2\varepsilon^2 + \ldots$ to find the expansion of the missing solution to (1.3).

C H A P T E R 2

Qualitative Modeling with Functions

It is often surprising that very simple mathematical modeling ideas can produce results with added value. Indeed, the solutions may be elegant and provide quality of understanding that obviates further exploration by more technical or complex means. In this chapter we explore a few simple approaches to qualitatively modeling phenomena with well-behaved functions.

2.1 MODELING SPECIES PROPAGATION

This problem concerns the factors that influence the number of species existing on an island. The discussion is adapted from [1]. One might speculate that factors affecting the number of species could include • Distance of the island from the mainland • Size of the island Of course limiting ourselves to these influences has the dual effect of making a tractable model that needs to be recognized as omitting many possible factors. The number of species may increase due to new species discovering the island as a suitable habitat. We will refer to this as the migration rate. Alternatively, species may become extinct due to competition. We will refer to this as the extinction rate. This discussion will be simplified by employing an aggregate total for the number of species and not attempting to distinguish the nature of each species, i.e., birds versus plants. Now we propose some basic modeling assumptions that appear reasonable.

The migration rate of new species decreases as the number of species on the island increases.

The argument for this is straight forward. The more species on an island the smaller the number of new species there is to migrate. See Figure 2.1 (a) for a qualitative picture.

The extinction rate of species increases as the number of species on the island increases.

Clearly the more species there are the more possibilities there are for species to die out. See Figure 2.1 (b) for a qualitative picture. If we plot the extinction rate and the migration rate on a single plot we identify the point of

intersection as an equilibrium, i.e., the migration is exactly offset by the extinction and the number of species on the island is a constant. We
13
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Number of species
Rate of Migration
M
(a) Migration curve.
Number of species
Rate of Extinction
E
(b) Extinction curve
FIGURE 2.1: Qualitative form of the migration and extinction curves.
will assume in this discussion that we are considering islands for which the number of species is roughly constant over time, i.e., they are in a state of equilibrium. Now we consider whether this simple model provides any added value. In particular, can it be used to address our questions posed at the outset. First, what is the effect of the distance of the island from the mainland on the number of bird species? One can characterize this effect by a shift in the migration curve. The further the island is away from the mainland, the less likely a species is to successfully migrate. Thus the migration curve is shifted down for far islands and shifted up for near islands. Presumably, this distance of the island from the mainland has no impact on the extinction curve. Thus, by examining the shift in the equilibrium, we may conclude that the number of species on an island decreases as the island's distance from the mainland increases. See Figure 2.2. Note in this model we assume that the time-scales are small enough that new species are not developed via evolution. While this may seem reasonable there is evidence that in some extreme climates, such as those found in the Galapagos Islands, variation may occur over shorter periods. There have been 140 different species of birds
2.2 SUPPLY AND DEMAND
In this section we sketch a well-known concept in economics, i.e., supply and demand. We shall see that relatively simple laws, when taken together, afford interesting insight into the relationship between producers and consumers. Furthermore, we may use this framework to make predictions such as • What

is the impact of a tax on the sale price? • What is the impact an increase in employees wages on sales price? Can the owner of the business pass this increase on to the consumer?
Law of Supply: An increase in the price of a commodity will result in an increase of the amount supplied.
Section 2.2 Supply and Demand 15
Number of species
E
M
Near Island
Far Island
FIGURE 2.2: The effect of distance of the island from the mainland is to shift the migration curve. Consequently the equilibrium solution dictates a smaller number of species will be supported for islands that are farther away from the mainland.
Law of Demand: If the price of a commodity increases, then the quantity demanded will decrease.
Thus, we may model the supply curve qualitatively by a monotonically increasing function. For simplicity we may assume a straight line with positive slope. Analogously, we may model the demand curve qualitatively by a monotonically decreasing function, which again we will take as a straight line. A flat demand curve may be interpreted as consumers being very sensitive to the price of a commodity. If the price goes up just a little, then the quantity in demand goes down significantly. Steep and flat supply and demand curves all have similar qualitative interpretations (see the problems).
2.2.1 Market Equilibrium
Given a supply curve and a demand curve we may plot them on the same axis and note their point of intersection $(q*,p*)$. This point is special for the following reason:
• The seller is willing to supply $q*$ at the price $p*$ • The demand is at the price $p*$ is $q*$ So both the supplier(s) and the purchaser(s) are happy economically speaking.
16 Chapter 2 Qualitative Modeling with Functions
Supply Curve
Quantity

Price
(a) Supply curve
Demand Curve
Quantity
Price
(b) Demand curve
FIGURE 2.3: (a) Qualitative form of supply and demand curves.

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1 0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
(q 1 ,p 1)(q 2 ,p 1)
(q 2 ,p 2)
(q 3 ,p 2)
D
S
o o
o o
FIGURE 2.4: The cobweb model illustrating a sequence of market adjustments.

2.2.2 Market Adjustment
Of course, in general markets do not exist in the perfect economic utopia described above. We may model the market adjustment as a sequence of points on the demand and supply curves. Based on market research it is estimated that consumers will demand a quantity q1 at a price p1. The supply and demand curves will permit a prediction of how the market will evolve. For

simplicity, we will assume that the initial point (q1,p1) is on the demand curve to the right of the equilibrium point. At the price p1 the supplier looks to his supply curve and proposes to sell a reduced quantity q2. Thus we move from right to left horizontally. Note that moving vertically to the supply curve would not make sense as this would correspond to offering the quantity q1 at an increased price. These goods will not sell at this price. From the point (q1,p2) the consumer will respond to the new reduced quantity q2 by being willing to pay more. This corresponds to moving vertically upward to the new point (q2,p2) on the supply curve. Now the supplier adjusts to the higher price being paid in the market place by increasing the quantity produced to q3. This process then continues, in theory, until an equilibrium is reached. It is possible that this will never happen, at least not without a basic adjustment to the shape of either the supply of demand curves, for example through cost cutting methods such as improved efficiency, or layoffs.

2.2.3 Taxation

The effect of a new tax on a product is to shift the demand curve down because consumers will not be willing to pay as much for the product (before the tax). Note that this leads to a new equilibrium point which reduces the price paid to the seller per item and reduces the quantity supplied by the producer. Thus one may conclude from this picture that the effect of a tax on alcohol is to reduce consumption as well as profit for the supplier. See Figure 2.5.

2.3 MODELING WITH PROPORTION AND SCALE

In the previous sections we have considered how simple functions may be employed to qualitatively model various situations and produce added value. Now we turn to considerations that assist in determining the nature of these functional dependencies in more complex terms.

2.3.1 Proportion

If a quantity y is proportional to a quantity x then we write

$y \propto x$

by which is meant

$y = kx$

for some constant of proportionality k.

Quantity (q)

Price (p)
D before tax
S
D after tax
(q old ,p old)
(q new ,p new)
Shift (amount of tax)
Cost burden to supplier
Cost burden to consumer
New cost to consumer
FIGURE 2.5: A tax corresponds to a downwards shift in the demand curve.

EXAMPLE 2.1
In 1678 Robert Hooke proposed that the restoring force F of a spring is proportional to its elongation e, i.e., $F \propto e$ or, $F = ke$
where k is the stiffness of the spring.
Note that the property of proportionality is symmetric, i.e.,
$y \propto x \to x \propto y$ (2.1)
and transitive, i.e.,
$y \propto x$ and $z \propto y \to z \propto x$ (2.2)
EXAMPLE 2.2
If $y = kx + b$ where k,b are constants, then
y 6∝ x
but
$y-b \propto x$
Inverse proportion. If $y \propto 1/x$ then y is said to be inversely proportional tox .
EXAMPLE 2.3
If y varies inversely as the square-root of x then
y =
k √x
Joint Variation. The volume of a cylinder is given by
$V = \pi r2h$
where r is the radius and h is the height. The volume is said to vary jointly with r2 and h, i.e., $V \propto r2$ and $V \propto h$

EXAMPLE 2.4
The volume of a given mass of gas is proportional to the temperature and inversely proportional to the pressure, i.e., $V \propto T$ and $V \propto 1/P$, or,
V = k
T P
EXAMPLE 2.5
Frictional drag due to the atmosphere is jointly proportional to the surface area S and the velocity v of the object.
Superposition of Proportions. Often a quantity will vary as the sum of proportions.
EXAMPLE 2.6
The stopping distance of a car when an emergency situation is encountered is the sum of the reaction time of the driver and the amount of time it takes for the breaks to dissipate the energy of the vehicle. The reaction distance is proportional to the velocity. The distance travelled once the breaks have been hit is proportional to the velocity squared. Thus,
stoppingdistance = k1v + k2v2
EXAMPLE 2.7
Numerical error in the computer estimation of the center difference formula for the derivative is given by e(h) = c1 h + c2h2 where the first term is due to roundoff error (finite precision) and the second term is due to truncation error. The value h is the distance δx in the definition of the derivative.
Direct Proportion. If
$y \propto x$ we say y varies in direct proportion to x. This is not true, for example, if $y \propto r2$. On the other hand, we may construct a direct proportion via the obvious change of variable x = r2. This simple trick always permits the investigation of the relationship between two variables such as this to be recast as a direct proportion.

−2 −1.5 −1 −0.5 0 0.5 1 1.5 2 0
0.2
0.4
0.6
0.8
1

1.2
1.4
1.6
1.8
2
y = r2/2
r
(a) Plot of y against r for y = r2/2.
0 0.5 1 1.5 2 2.5 3 3.5 4 0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
r2
y = r2/2
Slope 1/2
(b) Plot of y = r2/2 against r2.
−2 −1.5 −1 −0.5 0 0.5 1 1.5 2 −1
0
1
2
3
4
5
6
y = kr(r−1)
r
(c) Plot of y against r for y = kr(r+1).
−1 0 1 2 3 4 5 6 −1

0
1
2
3
4
5
6
r(r−1)
y = kr(r−1)
slope k
(d) Plot of y = kr(r+1) againstr(r+1)
FIGURE 2.6: Simple examples of how a proportion may be converted to a direct proportion.

2.3.2 Scale
Now we explore how the size of an object can be represented by an appropriate length scale if we restrict our attention to replicas that are geometrically similar. For example, a rectangle with sides l1 and w1 is geometrically similar to a rectangle with sides l2 and w2 if l1 l2 = w1 w2 = k (2.3) As the ratio κ = l1/w1 characterizes the geometry of the rectangle it is referred to as the shape factor. If two objects are geometrically similar, then it can be shown that they have the same shape factor. This follows directly from multiplying Equation (2.3) by the factor l2/w1, i.e.,
l1 w1
=
l2 w2
= k
l2 w1
Characteristic Length. Characteristic length is useful concept for characterizing a family of geometrically similar objects. We demonstrate this with an example. Consider the area of a rectangle of side l and width w where l and w may vary under the restriction that the resulting rectangle be geometrically similar to the rectangle with length l1 and width w1. An expression for the area of the varying triangle can be simplified as a consequence of the constraint imposed by geometric similarity. To see this

$A = lw$
$= l($
$w1l\ l1$
$)$
$= \kappa l2$
where $\kappa = w1/l1$, i.e., the shape factor. See Figure 2.7 for examples of characteristic lengths for the rectangle.
EXAMPLE 2.8
Watering a farmer's rectangular field requires an amount of area proportional to the area of the field. If the characteristic length of the field is doubled, how much additional water q will be needed, assuming the new field is geometrically similar to the old field? Solution: $q \propto l2$, i.e., $q = kl2$. Hence $q1 =$
$kl2\ 1$
$q2 = kl2\ 2$
Taking the ratio produces
$q1\ q2$
$=$
$l2\ 1\ l2\ 2$

h
w
d
FIGURE 2.7: The height l1, the width l2 and the diagonal l3 are all characteristic lengths for the rectangle.
Now if $q2 = 100$ acre feet of water are sufficient for a field of length $l2 = 100$, how much water will be required for a field of length $l1 = 200$? Sol.
$q1 = q2$
$l2\ 1\ l2\ 2$
$= 100$
$2002\ 1002$
$= 400$acrefeet ————————————
EXAMPLE 2.9
Why are gymnasts typically short? It seems plausible that the ability A, or natural talent, of gymnast would be proportional to strength and inversely proportional to weight, i.e., $A \propto$ strength and $A \propto 1$ weight

and taken jointly
A $\propto$
strength weight
One model for strength is that the strength of a limb is proportional to the crosssectional area of the muscle. The weight is proportional to the volume (assuming constant density of the gynmast). Now, assuming all gymnasts are geometrically similar with characteristic length l strength $\propto$ muscle area $\propto$ l2
Section 2.3 Modeling with Proportion and Scale 25
and
weight $\propto$ volume $\propto$ l3
so the ability A follows
A $\propto$
l2 l3 $\propto$
1 l So shortness equates to a talent for gymnastics. This problem was originally introduced in [2]. ————————————
EXAMPLE 2.10
Proportions and terminal velocity. Consider a uniform density spherical object falling under the influence of gravity. The object will travel will constant (terminal) velocity if the accelerating force due to gravity Fg = mg is balance exactly by the decelerating force due to atmospheric friction Fd = kSv2; S is the cross-sectional surface area and v is the velocity of the falling object. Our equilibrium condition is then Fg = Fd Since surface area satisfies S $\propto$ l2 it follows l $\propto$ S1/2. Given uniform densitym $\propto$ w $\propto$ l3 so it follows l $\propto$ m1/3. Combining proportionalities m1/3 $\propto$ S1/2 from which it follows by substitution into the force equation that m $\propto$ m2/3v2 or, after simplifying, v $\propto$ m1/6 ————————————
EXAMPLE 2.11
In this example we will attempt to model observed data displayed in Table 2.1 that relates the heart rate of mammals to there body weight. From the table we see that we would like to relate the heart rate as a function of body weight. Smaller animals have a faster heart rate than larger ones. But how do we estimate this proportionality? We begin by assuming that all the energy E produced by the body is used to maintain heat loss to the environment. This heat loss is in turn proportional to the surface area s of the body. Thus, E $\propto$ s The energy available to the body is produced by the process of respiration

and is assumed to be proportional to the oxygen available which is in turn proportional

mammal body weight (g) pulse rate shrew2 3.5 782 pipistrelle bat1 4 660 bat2 6 588 mouse1 25 670 hamster2 103 347 kitten2 117 300 rat1 200 420 rat2 252 352 guinea pig1 300 300 guinea pig1 437 269 rabbit2 1,340 251 rabbit1 2,000 205 oppossum2 2,700 187 little dog1 5,000 120 seal2 22,500 100 big dog1 30,000 85 goat2 33,000 81 sheep1 50,000 70 human1 70,000 72 swine2 100,000 70 horse2 415,000 45 horse1 450,000 38 ox1 500,000 40 elephant1 3,000,000 48

TABLE 2.1: Superscript 1 data source A.J. Clark; superscript 2 data source Altman and Dittmer. See also [1] and [2].

to the blood flow B through the lungs. Hence, $B \propto s$ If we denote the pulse rate as r we may assume $B \propto rV$ where V is the volume of the heart. We still need to incorporate the body weight w into this model. If we take W to be the weight of the heart assuming constant density of the heart it follows $W \propto V$ Also, if the bodies are assumed to be geometrically similar then $w \propto W$ so by transitivity $w \propto V$ and hence $B \propto rw$ Using the geometric similarity again we can relate the body surface area s to its weight w. From characteristic length scale arguments $v1/3 \propto s1/2$ so $s \propto w2/3$ from which we have $rw \propto w2/3$ or $r = kw-1/3$

To validate this model we plot w−1/3 versus r for the data Table 2.8. We see that for the larger animals with slower heart rates that this data appears linear and suggests this rather crude model actually is supported by the data. For much smaller animals there appear to be factors that this model is not capturing and the data falls off the line.

2.4 DIMENSIONAL ANALYSIS

In this chapter we have explored modeling with functions and proportion. In some instances, such as the mammalian heart rate, it is possible to cobble enough information together to actually extract a model; in particular, to identify the functional form for the relationship between the dependent and independent variables. Now we turn to a surprisingly powerful and simple tool known as dimensional analysis1. Dimensional analysis operates on the premise that equations contain terms that have units of measurement and that the validity of these equations, or laws, are not dependent on the system of measurement. Rather these equations relate variables that have inherent physical dimensions that are derived from the fundamental dimensions of mass, length and time. We label these dimensions generically as M,L and T, respectively. As we shall see, dimensional analysis provides an effective tool for mathematical modeling in many situations. In particular, some benefits include

1This dimension should not be confused with the usual notion of geometric dimension.

0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0
100
200
300
400
500
600
700
800
pulse rate r
weight $w^{-1/3}$
mammalian heart rate model

FIGURE 2.8: Testing the model produced by proportionality. For the model to fit, the data should sit on a straight line eminating from the origin.

• determination of the form of a joint proportion • reduce number of variables in a model • enforcement of dimensional consistency • ability to study scaled versions of models

2.4.1 Dimensional homogeneity

An equation is said to be dimensionally homogeneous if all the terms in the equation have the same physical dimension.

EXAMPLE 2.12

All the laws of physics are dimensionally homogeneous. Consider Newton's law

$F = ma$

The units on the right side are

$M \cdot$

$L\ T2$ so we conclude that the physical dimension of a force must be MLT^{-2}. ——

EXAMPLE 2.13

The equation of motion of a linear spring with no damping is

m

d2x dt2

$+ kx = 0$

What are the units of the spring constant? Dimensionally we can recast this equation as $MLT^{-2} + M^aL^bT^cL = 0$ Matching exponents for each dimension permits the calculation of a,b and c.

$M : a = 1$ $L : 1 = b + 1$ $T : -2 = c$ Thus we conclude that the spring constant has the dimensions MT^{-2}. ————————————

EXAMPLE 2.14

Let v be velocity, t be time and x be distance. The model equation $v2 = t2 + x\ t$ is dimensionally inconsistent.

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EXAMPLE 2.15

An angle may be defined by the formula

$\theta =$

s r where the arclength s subtends the angle θ and r is the radius of the circle. Clearly this angle is dimensionless.

2.4.2 Discovering Joint Proportions

If in the formulation of a problem we are able to identify a dependent and one or more independent variables, it is often possible to identify the form of a joint proportion. The form of the proportion is actually constrained by the fact that the equations must be dimensionally consistent.

EXAMPLE 2.16 Drag Force on an Airplane

In this problem we consider the drag force FD on an airplane. As our model we propose that this drag force (dependent variable) is proportional to the independent variables • cross-sectional area A of airplane • velocity v of airplane • density ρ of the air As a joint proportion we have $FD = kA^av^b\rho^c$ where a,b and c are unknown exponents. As a consequence of dimensional consistency we have $MLT^{-2} = (M^0L^0T^0)(L^2)^a(L\ T)^b(M\ L3)^c = M^cL^{2a-3c+b}T^{-b}$

From the M exponent we conclude $c = 1$. From the T exponent $b = 2$ and from the L exponent it follows that $1 = 2a-3c + b$, whence $a = 1$. Thus the only possibility for the form of this joint proportion is $F_D = kAv^2\rho$

Note that if the density of were a constant it would be appropriate to simplify this dependency as $F = \tilde{k}Av^2$ but now the constant $\tilde{k}$ actually has dimensions. ———————————

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2.4.3 Procedure for Nondimensionalization

Consider the nonlinear model for a pendulum

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

Based on the terms in this model we may express the solution very generally as a relationship between these included terms, i.e.,

$$\phi(\theta,g,l,t) = 0$$

Note that the angle in this model is dimensionless but the other variables all have dimensions. We can convert this equation into a new equation where none of the terms have dimensions. This will be referred to, for obvious reasons, as a dimensional form of the model. To accomplish this, let $\tau = t\sqrt{l/g}$.

The substitution of variables may be accomplished by noting that

$$\frac{d^2\theta}{dt^2} = \frac{d^2\theta}{\frac{l}{g}d\tau^2}$$

Thus, after cancelation, the dimensionless form for the nonlinear pendulum model is $\frac{d^2\theta}{d\tau^2} = -\sin\theta$ Now the solution has the general form

$$f(\theta,\tau) = 0,$$

or equivalently,

$$f\left(\theta,\sqrt{\frac{l}{g}}\,t\right) = 0$$

This is a special case of a more general theory. The Buckingham π-theorem. Any dimensionally homogeneous equation with physical variables $x_1,...,x_m$ expressed

$$\phi(x_1,...,x_m)$$

may be rewritten in terms of its associated dimensionless variables π1,...,πn as

f(π1,...,πn) = 0

where

πk = xak1 1 ...xakm m

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2.4.4 Modeling with Dimensional Analysis

Now we consider two examples of the application of the ideas described above concerning dimensional analysis. In each of these examples there is more than one dimensionless parameter and it is appropriate to apply the Buckingham π-theorem.

The Pendulum. In this example the goal is to understand how the period of a pendulum depends on the other parameters that describe the nature of the pendulum. The first task is to identify this set of parameters that act as the independent variables on which the period P depends. Obvious candidates include From this list we are motivated to write

variable symbol dimensions mass m M length l L gravity g LT−2 angle θ0 M0L0T0 period P T

TABLE 2.2: Parameters influencing the motion of a simple pendulum.

P = ϕ(m,l,g,θ0)

As we shall see, attempting to establish the form of ϕ directly is unnecessarily complicated. Instead, we pursue the idea of dimensional analysis. To begin this modeling procedure, we compute the values of a,b,c,d and e that make the quantity π = malbgcθd 0Pe a dimensionless parameter. Again, this is done by equating exponents on the fundamental dimensions

M0L0T0 = MaLb(LT−2)c(M0L0T0)dTe

From M0: 0 = a. From L0: 0 = b + c. From T0: 0 = −2c + e. From this we may conclude that

π = m0l−cgcθd 0P2c

or, after collecting terms,

π = θd 0

gP2 l cwhere π is dimensionless for any values of d and c. Thus we have found a complete set of dimensionless parameters π1 = θ0

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and

$\pi 2$ =rg l
P
Since the period P of the pendulum is based on dimensionally consistent physical laws we may apply the Buckingham π-theorem. In general,
$f(\pi 1,\pi 2) = 0$
which we rewrite as
$\pi 2 = h(\pi 1)$
which now becomes
P =sl g
$h(\theta 0)$
We may draw two immediate conclusions from this model. • The period depends on the square root of the length of the pendulum. • The period is independent of the mass Of course we have not really shown these conclusions to be "true". But now we have something to look for that can be tested. We could test these assertions and if they contradict our model then we would conclude that we are missing an important factor that governs the period of the pendulum. Indeed, as we have neglected drag forces due to friction it seems our model will have limited validity. The functional form of h may now be reasonable calculated as there is only one independent variable θ0. If we select several different initial displacements θ0(i) and measure the period for each one we have a set of domain–range values h(θ0(i)) = Pirg l to which a data fitting procedure may now be applied.
The damped pendulum. We assumed that there was no damping of this pendulum above due to air resistance. We can include a drag force FD by augmenting the list of relevant parameters to
m,l,g,θ0,P,FD
Now our dimensionless parameter takes the form
π = malbgcθd 0PeFf D
Converting to dimensions
M0L0T0 = MaLb(LT−2)c(M0L0T0)dTe(MLT−2)f
As
$0 = a + f$
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it is no longer possible to immediately conclude that a = 0. In fact, it is not. (See problems).

Fluid Flow. Consider the parameters governing the motion of an oil past a spherical ball bearing. Let's assume they include:

variable symbol dimensions velocity v LT^{-1} density ρ ML^{-3} gravity g LT^{-2} radius l L viscosity μ $ML^{-1}T^{-1}$

TABLE 2.3: Parameters influencing the motion of a fluid around a submerged body.

The dimensionless combination has the form $\pi = v^a \rho^b l^c g^d \mu^e$

Using the explicit form of the physical dimensions for each term we have

$M^0L^0T^0 = (LT^{-1})^a(ML^{-3})^b(L)^c(LT^{-2})^d(ML^{-1}T^{-1})^e$

Again, matching exponents

M : $0 = b + e$ L : $0 = a-3b + c + d-e$ T : $0 = -a-2d-e$ Sinc there are three equations and five unknowns the system is said to be undetermined. Given these numbers, we anticipate that there we can solve for three variable in terms of the other two. Of course, we can solve in terms of any of the two variables. For example, $a = -2d-e$ $b = -e$ $c = d-e$ Plugging these constraints into our expression for π gives $\pi = \frac{v^2}{lg}^{-d} \frac{\rho l v}{\mu}^{-e}$ Thus, our two dimensionless parameters are the Froude number

$\pi_1 =$

$\frac{v^2}{lg}$

and the Reynolds number

$\pi_2 =$

$\frac{v \rho l}{\mu}$ For further discussion see Giordano, Wells and Wilde, UMAP module 526.

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PROBLEMS 2.1. By drawing a new graph, show the effect of the size of the island on the • extinction curve • migration curve Now predict how island size impacts the number of species on the island. Does this seem reasonable? 2.2. Give an example of a commodity that does not obey the • law of supply • law of demand and justify your claim. 2.3. Translate into words the qualitative interpretation of the slope of the supply and demand curves. In particular, what is the meaning of a • flat supply curve? • steep supply curve? • steep demand curve? 2.4. Consider the table of market adjustments below. Assuming the first point is on the demand curve, compute the equations of both the demand and supply curve. Using these equations, find the missing values A,B,C,D. What is the equilibrium point? Do you think the market will adjust to it?

quantity price 3 0.7 0.14 0.7 0.14 0.986 0.1972 0.986 A =? B =? C =? D =?

2.5. Using the cobweb plot show an example of a market adjustment that oscillates wildly out of control. Can you describe a qualitative feature of the supply and demand curves that will ensure convergence to an equilibrium? 2.6. Consider the effect of a price increase on airplane fuel (kerosene) on the airline industry. What effect does this have on the supply curve? Will the airline industry be able to pass this cost onto the flying public? How does your answer differ if the demand curve is flat versus steep? 2.7. Prove properties 2.1 and 2.2. 2.8. Is the temperature measured in degrees Fahrenheit proportional to the temperature measured in degrees centigrade? 2.9. Consider the Example 2.6 again. Demonstrate the proportionalities stated. For the case of the breaking distance equate the work done by the breaks to the dissipated kinetic energy of the car. 2.10. Items at the grocery store typically come in various sizes and the cost per unit is generally smaller for larger items. Model the cost per unit weight by considering the superposition of proportions due to the costs of • production • packaging

• shipping the product. What predictions can you make from this model. This problem was adapted from Bender [1]. 2.11. Go to your nearest supermarket and collect data on the cost of items as a function of size. Do these data behave in a fashion predicted by your model in the previous problem? 2.12. In this problem take the diagonal of a rectangle as it's length scale l. Show by direct calculation that this can be used to measure the area, i.e.,

$$A = \alpha l2$$

Determine the constant of proportionality α in terms of the shape factor of the rectangle. 2.13. Consider a radiator designed as a spherical shell. If the characteristic length of the shell doubles (assume the larger radiator is geometrically similar to the smaller radiator) what is the effect on the amount of heat loss? What if the design of the radiator is a parallelpiped instead? 2.14. How does the argument in Example 2.10 change if the falling object is not spherical but some other irregular shape? 2.15. Extend the definition of geometric similarity for • parallelpipeds • irregularly shaped objects Can you propose a computer algorithm for testing whether two objects are geomtrically similar? 2.16. Consider the force on a pendulum due to air friction modeled by

$F_D = \kappa v^2$

Determine the units of κ. 2.17. Newton's law of gravitation states that

$$F = \frac{G m_1 m_2}{r^2}$$

where F is the force between two objects of masses m_1, m_2 and r is the distance between them. (a) What is the physical dimension of G? (b) Compute two dimensionless products π_1 and π_2 and show explicitly that they satisfy the Buckingham π-theorem. 2.18. This problem concerns the pendulum example described in subsection 2.4.4. Repeat the analysis to determine the dimensionless parameter(s) but now omit the gravity term g. Discuss. 2.19. This problem concerns the pendulum example described in subsection 2.4.4. Repeat the analysis for determining all the dimensionless parameters but now include a parameter κ associated with the drag force of the form $F_D = \kappa v$. Hint: first compute the dimensions of κ. 2.20. Convert the equation governing the distance travelled by a projectile,

$$\frac{d^2x}{dt^2} = -\frac{gR^2}{(x + R)^2},$$

to the form

$$\frac{d^2y}{d\tau^2} = -\frac{1}{(y + 1)^2},$$

where y and τ are dimensionless. 2.21. Reconsider the example in subsection 2.4.4. Instead of solving for a,b,c in terms of d,e solve for c,d,e in terms of a and b. Show that now

$$\pi'_1 = \frac{v}{\sqrt{lg}}$$

and

π'

2 =
ρl3/2g1/2 μ
Show also that both π
′ 1 and π
′ 2 can be written in terms of π1 and π2. 2.22. Consider an object with surface area A traveling with a velocity v through a medium with kinematic viscosity μ and density ρ. (a) Assuming the effect of μ is small compute the drag force due to the density Fρ. (b) Assuming the effect of ρ is small compute the drag force due to the kinematic viscosity Fμ. c) Compute the dimensionless ratio of these drag forces and discuss what predictions you can make. 2.23. Assume a drag force of the form Fd = κv2
acts on a pendulum in addition to the gravity force. Use dimensional analysis to show that the solution of the pendulum equation can be written in the form θ = ψ(tpl/g,lκ/m).2.24. How does the required power P of a helicopter engine depend on the length of the rotors l? The rotors are pushing air so presumable the density ρ as well as the weight of the helicopter w = mg are variables that affect the power requirement. Draw a sketch of your result plotting P versus l. See [3] for more discussion of this problem.
Bibliography

Week 9 and 10

Precursors[edit]
Discussions on the mathematics of games began long before the rise of modern mathematical game theory. Cardano's work on games of chance in Liber de ludo aleae (Book on Games of Chance), which was written around 1564 but published posthumously in 1663, formulated some of the field's basic ideas. In the 1650s, Pascal and Huygens developed the concept of expectation on reasoning about the structure of games of chance, and Huygens published his gambling calculus in De ratiociniis in ludo aleæ (On Reasoning in Games of Chance) in 1657.

In 1713, a letter attributed to Charles Waldegrave analyzed a game called "le Her". He was an active Jacobite and uncle to James Waldegrave, a British diplomat.[2][3] In this letter, Waldegrave provided a minimax mixed strategy solution to a two-person version of the card game le Her, and the problem is now known as Waldegrave problem. In his 1838 Recherches sur les principes mathématiques de la théorie des richesses (Researches into the Mathematical Principles of the Theory of Wealth), Antoine Augustin Cournot considered a duopoly and presented a solution that is the Nash equilibrium of the game.
In 1913, Ernst Zermelo published Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels (On an Application of Set Theory to the Theory of the Game of Chess), which proved that the optimal chess strategy is strictly determined. This paved the way for more general theorems.[4]
In 1938, the Danish mathematical economist Frederik Zeuthen proved that the mathematical model had a winning strategy by using Brouwer's fixed point theorem.[5] In his 1938 book Applications aux Jeux de Hasard and earlier notes, Émile Borel proved a minimax theorem for two-person zero-sum matrix games only when the pay-off matrix is symmetric and provided a solution to a non-trivial infinite game (known in English as Blotto game). Borel conjectured the non-existence of mixed-strategy equilibria in finite two-person zero-sum games, a conjecture that was proved false by von Neumann.
Birth and early developments[edit]

John von Neumann
Game theory did not exist as a unique field until John von Neumann published the paper On the Theory of Games of Strategy in 1928.[6][7] Von Neumann's original proof used Brouwer's fixed-point theorem on continuous mappings into compact convex sets, which became a standard method in game theory

and mathematical economics. His paper was followed by his 1944 book Theory of Games and Economic Behavior co-authored with Oskar Morgenstern.[8] The second edition of this book provided an axiomatic theory of utility, which reincarnated Daniel Bernoulli's old theory of utility (of money) as an independent discipline. Von Neumann's work in game theory culminated in this 1944 book. This foundational work contains the method for finding mutually consistent solutions for two-person zero-sum games. Subsequent work focused primarily on cooperative game theory, which analyzes optimal strategies for groups of individuals, presuming that they can enforce agreements between them about proper strategies.[9]

John Nash

In 1950, the first mathematical discussion of the prisoner's dilemma appeared, and an experiment was undertaken by notable mathematicians Merrill M. Flood and Melvin Dresher, as part of the RAND Corporation's investigations into game theory. RAND pursued the studies because of possible applications to global nuclear strategy.[10] Around this same time, John Nash developed a criterion for mutual consistency of players' strategies known as the Nash equilibrium, applicable to a wider variety of games than the criterion proposed by von Neumann and Morgenstern. Nash proved that every finite n-player, non-zero-sum (not just two-player zero-sum) non-cooperative game has what is now known as a Nash equilibrium in mixed strategies.

Game theory experienced a flurry of activity in the 1950s, during which the concepts of the core, the extensive form game, fictitious play, repeated games, and the Shapley value were developed. The 1950s also saw the first applications of game theory to philosophy and political science.

Prize-winning achievements[edit]

In 1965, Reinhard Selten introduced his solution concept of subgame perfect equilibria, which further refined the Nash equilibrium. Later he would introduce trembling hand perfection as well. In 1994 Nash, Selten and Harsanyi became Economics Nobel Laureates for their contributions to economic game theory.

In the 1970s, game theory was extensively applied in biology, largely as a result of the work of John Maynard Smith and his evolutionarily stable strategy. In addition, the concepts of correlated equilibrium, trembling hand perfection, and common knowledge[a] were introduced and analyzed.

In 2005, game theorists Thomas Schelling and Robert Aumann followed Nash, Selten, and Harsanyi as Nobel Laureates. Schelling worked on dynamic models, early examples of evolutionary game theory. Aumann contributed more to the equilibrium school, introducing equilibrium coarsening and correlated equilibria, and developing an extensive formal analysis of the assumption of common knowledge and of its consequences.

In 2007, Leonid Hurwicz, Eric Maskin, and Roger Myerson were awarded the Nobel Prize in Economics "for having laid the foundations of mechanism design theory". Myerson's contributions include the notion of proper equilibrium, and an important graduate text: Game Theory, Analysis of Conflict.[1] Hurwicz introduced and formalized the concept of incentive compatibility.

In 2012, Alvin E. Roth and Lloyd S. Shapley were awarded the Nobel Prize in Economics "for the theory of stable allocations and the practice of market design". In 2014, the Nobel went to game theorist Jean Tirole.

Game types[edit]

See also: List of games in game theory

Cooperative / non-cooperative[edit]

Main articles: Cooperative game theory and Non-cooperative game

A game is cooperative if the players are able to form binding commitments externally enforced (e.g. through contract law). A game is non-cooperative if players cannot form alliances or if all agreements need to be self-enforcing (e.g. through credible threats).[11]

Cooperative games are often analyzed through the framework of cooperative game theory, which focuses on predicting which coalitions will form, the joint actions that groups take, and the resulting collective payoffs. It is opposed to the traditional non-cooperative game theory which focuses on predicting individual players' actions and payoffs and analyzing Nash equilibria.[12][13] The focus on individual payoff can result in a phenomenon known as Tragedy of the Commons, where resources are used to a collectively inefficient level. The lack of formal negotiation leads to the deterioration of public goods through over-use and under provision that stems from private incentives.[14] Cooperative game theory provides a high-level approach as it describes only the structure, strategies, and payoffs of coalitions, whereas non-cooperative game theory also looks at how bargaining procedures will affect the distribution of payoffs within each coalition. As non-cooperative game theory is more general, cooperative games can be analyzed through the approach of non-cooperative game theory (the converse does not hold) provided that sufficient assumptions are made to encompass all the possible strategies available to players due to the possibility of external enforcement of cooperation. While using a single theory may be desirable, in many instances insufficient information is available to accurately model the formal procedures available during the strategic bargaining process, or the resulting model would be too complex to offer a practical tool in the real world. In such cases, cooperative game theory provides a simplified approach that allows analysis of the game at large without having to make any assumption about bargaining powers.

Symmetric / asymmetric[edit]

Main article: Symmetric game

A symmetric game is a game where the payoffs for playing a particular strategy depend only on the other strategies employed, not on who is playing them. That is, if the identities of the players can be changed without changing the payoff to the strategies, then a game is symmetric. Many of the commonly studied 2×2 games are symmetric. The standard representations of chicken, the prisoner's dilemma, and the stag hunt are all symmetric games. Some[who?] scholars would consider certain asymmetric games as examples of these games as well. However, the most common payoffs for each of these games are symmetric.

	E	F
E	1, 2	0, 0
F	0, 0	1, 2

An asymmetric game

The most commonly studied asymmetric games are games where there are not identical strategy sets for both players. For instance, the ultimatum game and similarly the dictator game have different strategies for each player. It is possible, however, for a game to have identical strategies for both players, yet be asymmetric. For example, the game pictured in this section's graphic is asymmetric despite having identical strategy sets for both players.

Zero-sum / non-zero-sum[edit]

Main article: Zero-sum game

	A	B
A	−1, 1	3, −3
B	0, 0	−2, 2

A zero-sum game

Zero-sum games (more generally, constant-sum games) are games in which choices by players can neither increase nor decrease the available resources. In zero-sum games, the total benefit goes to all players in a game, for every combination of strategies, always adds to zero (more informally, a player benefits only at the equal expense of others).[15] Poker exemplifies a zero-sum game (ignoring the possibility of the house's cut), because one wins exactly the amount one's opponents lose. Other zero-sum games include matching pennies and most classical board games including Go and chess.

Many games studied by game theorists (including the famed prisoner's dilemma) are non-zero-sum games, because the outcome has net results greater or less than zero. Informally, in non-zero-sum games, a gain by one player does not necessarily correspond with a loss by another.

Constant-sum games correspond to activities like theft and gambling, but not to the fundamental economic situation in which there are potential gains from trade. It is possible to transform any constant-sum game into a (possibly asymmetric) zero-sum game by adding a dummy player (often called "the board") whose losses compensate the players' net winnings.

Simultaneous / sequential[edit]

Main articles: Simultaneous game and Sequential game

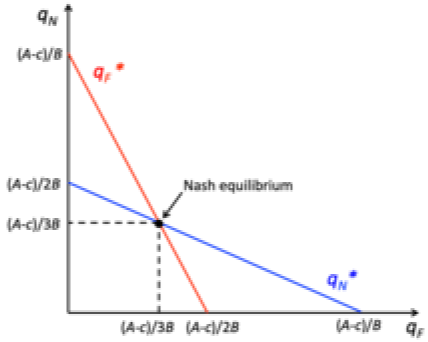
Simultaneous games are games where both players move simultaneously, or instead the later players are unaware of the earlier players' actions (making them effectively simultaneous). Sequential games (or dynamic games) are games where later players have some knowledge about earlier actions. This need not be perfect information about every action of earlier players; it might be very little knowledge. For instance, a player may know that an earlier

player did not perform one particular action, while they do not know which of the other available actions the first player actually performed.
The difference between simultaneous and sequential games is captured in the different representations discussed above. Often, normal form is used to represent simultaneous games, while extensive form is used to represent sequential ones. The transformation of extensive to normal form is one way, meaning that multiple extensive form games correspond to the same normal form. Consequently, notions of equilibrium for simultaneous games are insufficient for reasoning about sequential games; see subgame perfection.
In short, the differences between sequential and simultaneous games are as follows:

	Sequential	Simultaneous
Normally denoted by	Decision trees	Payoff matrices
Prior knowledge of opponent's move?	Yes	No
Time axis?	Yes	No
Also known as	Extensive-form game Extensive game	Strategy game Strategic game

Cournot Competition[edit]
The Cournot competition model involves players choosing quantity of a homogenous product to produce independently and simultaneously, where marginal cost can be different for each firm and the firm's payoff is profit. The production costs are public information and the firm aims to find their profit-maximising quantity based on what they believe the other firm will produce and behave like monopolies. In this game firms want to produce at the monopoly quantity but there is a high incentive to deviate and produce more, which decreases the market-clearing price.[16] For example, firms may be tempted to deviate from the monopoly quantity if there is a low monopoly quantity and high price, with the aim of increasing production to maximise profit.[16] However this option does not provide the highest payoff, as a firm's ability to maximise profits depends on its market share and the elasticity of the market demand.[17] The Cournot equilibrium is reached when each firm operates on their reaction function with no incentive to deviate, as they have the best response based on the other firms output.[16]

Within the game, firms reach the Nash equilibrium when the Cournot equilibrium is achieved.



Equilibrium for Cournot quantity competition

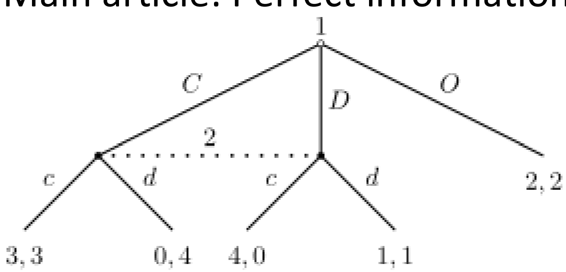
Bertrand Competition[edit]

Main article: Bertrand competition

The Bertrand competition, assumes homogenous products and a constant marginal cost and players choose the prices.[16] The equilibrium of price competition is where the price is equal to marginal costs, assuming complete information about the competitors' costs. Therefore, the firms have an incentive to deviate from the equilibrium because a homogenous product with a lower price will gain all of the market share, known as a cost advantage.[18]

Perfect information and imperfect information[edit]

Main article: Perfect information



A game of imperfect information (the dotted line represents ignorance on the part of player 2, formally called an information set)

An important subset of sequential games consists of games of perfect information. A game is one of perfect information if all players, at every move in the game, know the moves previously made by all other players. In reality, this can be applied to firms and consumers having information about price and quality of all the available goods in a market.[19] An imperfect information game is played when the players do not know all moves already made by the opponent such as a simultaneous move game.[16] Most games studied in game theory are imperfect-information games.[citation needed]

Examples of perfect-information games include tic-tac-toe, checkers, chess, and Go.[20][21][22][23]
Many card games are games of imperfect information, such as poker and bridge.[24] Perfect information is often confused with complete information, which is a similar concept.[citation needed] Complete information requires that every player know the strategies and payoffs available to the other players but not necessarily the actions taken, whereas perfect information is knowledge of all aspects of the game and players.[25] Games of incomplete information can be reduced, however, to games of imperfect information by introducing "moves by nature".[26]

Bayesian game[edit]

Main article: Bayesian game

One of the assumptions of the Nash equilibrium is that every player has correct beliefs about the actions of the other players. However, there are many situations in game theory where participants do not fully understand the characteristics of their opponents. Negotiators may be unaware of their opponent's valuation of the object of negotiation, companies may be unaware of their opponent's cost functions, combatants may be unaware of their opponent's strengths, and jurors may be unaware of their colleague's interpretation of the evidence at trial. In some cases, participants may know the character of their opponent well, but may not know how well their opponent knows his or her own character.[27]
Bayesian game means a strategic game with incomplete information. For a strategic game, decision makers are players, and every player has a group of actions. A core part of the imperfect information specification is the set of states. Every state completely describes a collection of characteristics relevant to the player such as their preferences and details about them. There must be a state for every set of features that some player believes may exist.[28]

	2 wishes to meet 1	
	B	S
B	2,1	0,0
S	0,0	1,2

	2 wishes to avoid 1	
	B	S
B	2,0	0,2
S	0,1	1,0

example of bayesian game

For example, where Player 1 is unsure whether Player 2 would rather date her or get away from her, while Player 2 understands Player 1's preferences as before. To be specific, supposing that Player 1 believes that Player 2 wants to date her under a probability of 1/2 and get away from her under a probability of 1/2 (this evaluation comes from Player 1's experience probably: she faces players who want to date her half of the time in such a case and players who want to avoid her half of the time). Due to the probability involved, the analysis of this situation requires to understand the player's preference for the draw, even though people are only interested in pure strategic equilibrium.

Combinatorial games[edit]

Games in which the difficulty of finding an optimal strategy stems from the multiplicity of possible moves are called combinatorial games. Examples include chess and Go. Games that involve imperfect information may also have a strong combinatorial character, for instance backgammon. There is no unified theory addressing combinatorial elements in games. There are, however, mathematical tools that can solve particular problems and answer general questions.[29]

Games of perfect information have been studied in combinatorial game theory, which has developed novel representations, e.g. surreal numbers, as well as combinatorial and algebraic (and sometimes non-constructive) proof methods to solve games of certain types, including "loopy" games that may result in infinitely long sequences of moves. These methods address games with higher combinatorial complexity than those usually considered in traditional (or "economic") game theory.[30][31] A typical game that has been solved this way is Hex. A related field of study, drawing from computational complexity theory, is game complexity, which is concerned with estimating the computational difficulty of finding optimal strategies.[32]

Research in artificial intelligence has addressed both perfect and imperfect information games that have very complex combinatorial structures (like chess, go, or backgammon) for which no provable optimal strategies have been found. The practical solutions involve computational heuristics, like alpha–beta pruning or use of artificial neural networks trained by reinforcement learning, which make games more tractable in computing practice.[29][33]

Infinitely long games[edit]

Main article: Determinacy

Games, as studied by economists and real-world game players, are generally finished in finitely many moves. Pure mathematicians are not so constrained, and set theorists in particular study games that last for infinitely many moves, with the winner (or other payoff) not known until after all those moves are completed.

The focus of attention is usually not so much on the best way to play such a game, but whether one player has a winning strategy. (It can be proven, using the axiom of choice, that there are games – even with perfect information and where the only outcomes are "win" or "lose" – for which neither player has a winning strategy.) The existence of such strategies, for cleverly designed games, has important consequences in descriptive set theory.

Discrete and continuous games[edit]

Much of game theory is concerned with finite, discrete games that have a finite number of players, moves, events, outcomes, etc. Many concepts can be extended, however. Continuous games allow players to choose a strategy from a continuous strategy set. For instance, Cournot competition is typically modeled with players' strategies being any non-negative quantities, including fractional quantities.

Differential games[edit]

Differential games such as the continuous pursuit and evasion game are continuous games where the evolution of the players' state variables is governed by differential equations. The problem of finding an optimal strategy in a differential game is closely related to the optimal control theory. In particular, there are two types of strategies: the open-loop strategies are found using the Pontryagin maximum principle while the closed-loop strategies are found using Bellman's Dynamic Programming method.

A particular case of differential games are the games with a random time horizon.[34] In such games, the terminal time is a random variable with a given probability distribution function. Therefore, the players maximize the mathematical expectation of the cost function. It was shown that the modified optimization problem can be reformulated as a discounted differential game over an infinite time interval.

Evolutionary game theory[edit]

Evolutionary game theory studies players who adjust their strategies over time according to rules that are not necessarily rational or farsighted.[35] In general, the evolution of strategies over time according to such rules is modeled as a Markov chain with a state variable such as the current strategy profile or how the game has been played in the recent past. Such rules may feature imitation, optimization, or survival of the fittest.

In biology, such models can represent evolution, in which offspring adopt their parents' strategies and parents who play more successful strategies (i.e. corresponding to higher payoffs) have a greater number of offspring. In the social sciences, such models typically represent strategic adjustment by players who play a game many times within their lifetime and, consciously or unconsciously, occasionally adjust their strategies.[36]

Stochastic outcomes (and relation to other fields)[edit]

Individual decision problems with stochastic outcomes are sometimes considered "one-player games". They may be modeled using similar tools within the related disciplines of decision theory, operations research, and areas of artificial intelligence, particularly AI planning (with uncertainty) and multi-agent system. Although these fields may have different motivators, the mathematics involved are substantially the same, e.g. using Markov decision processes (MDP).[37]

Stochastic outcomes can also be modeled in terms of game theory by adding a randomly acting player who makes "chance moves" ("moves by nature").[38] This player is not typically considered a third player in what is otherwise a two-player game, but merely serves to provide a roll of the dice where required by the game.

For some problems, different approaches to modeling stochastic outcomes may lead to different solutions. For example, the difference in approach between MDPs and the minimax solution is that the latter considers the worst-case over a set of adversarial moves, rather than reasoning in expectation about these moves given a fixed probability distribution. The minimax approach may be advantageous where stochastic models of uncertainty are not available, but may also be overestimating extremely unlikely (but costly) events, dramatically swaying the strategy in such scenarios if it is assumed that an adversary can force such an event to happen.[39] (See Black swan theory for more discussion on this kind of modeling issue, particularly as it relates to predicting and limiting losses in investment banking.)

General models that include all elements of stochastic outcomes, adversaries, and partial or noisy observability (of moves by other players) have also been studied. The "gold standard" is considered to be partially observable stochastic game (POSG), but few realistic problems are computationally feasible in POSG representation.[39]

Metagames[edit]

These are games the play of which is the development of the rules for another game, the target or subject game. Metagames seek to maximize the utility value of the rule set developed. The theory of metagames is related to mechanism design theory.

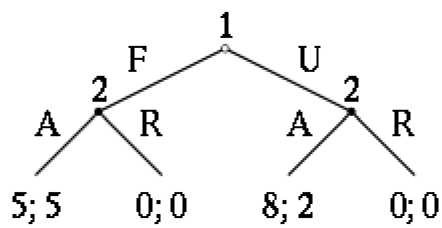
The term metagame analysis is also used to refer to a practical approach developed by Nigel Howard,[40] whereby a situation is framed as a strategic game in which stakeholders try to realize their objectives by means of the options available to them. Subsequent developments have led to the formulation of confrontation analysis.

Pooling games[edit]

These are games prevailing over all forms of society. Pooling games are repeated plays with changing payoff table in general over an experienced path, and their equilibrium strategies usually take a form of evolutionary social convention and economic convention. Pooling game theory emerges to formally recognize the interaction between optimal choice in one play and the emergence of forthcoming payoff table update path, identify the invariance existence and robustness, and predict variance over time. The theory is based upon topological transformation classification of payoff table update over time to predict variance and invariance, and is also within the jurisdiction of the computational law of reachable optimality for ordered system.[41]

Mean field game theory[edit]

Mean field game theory is the study of strategic decision making in very large populations of small interacting agents. This class of problems was considered in the economics literature by Boyan Jovanovic and Robert W. Rosenthal, in the engineering literature by Peter E. Caines, and by mathematicians Pierre-Louis Lions and Jean-Michel Lasry.

Representation of games[edit]
The games studied in game theory are well-defined mathematical objects. To be fully defined, a game must specify the following elements: the players of the game, the information and actions available to each player at each decision point, and the payoffs for each outcome. (Eric Rasmusen refers to these four "essential elements" by the acronym "PAPI".)[42][43][44][45] A game theorist typically uses these elements, along with a solution concept of their choosing, to deduce a set of equilibrium strategies for each player such that, when these strategies are employed, no player can profit by unilaterally deviating from their strategy. These equilibrium strategies determine an equilibrium to the game—a stable state in which either one outcome occurs or a set of outcomes occur with known probability.
Most cooperative games are presented in the characteristic function form, while the extensive and the normal forms are used to define noncooperative games.
Extensive form[edit]
Main article: Extensive form game


An extensive form game
The extensive form can be used to formalize games with a time sequencing of moves. Games here are played on trees (as pictured here). Here each vertex (or node) represents a point of choice for a player. The player is specified by a number listed by the vertex. The lines out of the vertex represent a possible action for that player. The payoffs are specified at the bottom of the tree. The extensive form can be viewed as a multi-player generalization of a decision tree.[46] To solve any extensive form game, backward induction must be used. It involves working backward up the game tree to determine what a rational player would do at the last vertex of the tree, what the player with the previous move would do given that the player with the last move is rational, and so on until the first vertex of the tree is reached.[47]
The game pictured consists of two players. The way this particular game is structured (i.e., with sequential decision making and perfect information),

Player 1 "moves" first by choosing either F or U (fair or unfair). Next in the sequence, Player 2, who has now seen Player 1's move, chooses to play either A or R. Once Player 2 has made their choice, the game is considered finished and each player gets their respective payoff. Suppose that Player 1 chooses U and then Player 2 chooses A: Player 1 then gets a payoff of "eight" (which in real-world terms can be interpreted in many ways, the simplest of which is in terms of money but could mean things such as eight days of vacation or eight countries conquered or even eight more opportunities to play the same game against other players) and Player 2 gets a payoff of "two".

The extensive form can also capture simultaneous-move games and games with imperfect information. To represent it, either a dotted line connects different vertices to represent them as being part of the same information set (i.e. the players do not know at which point they are), or a closed line is drawn around them. (See example in the imperfect information section.)

Normal form[edit]

Main article: Normal-form game

	Player 2 chooses Left	Player 2 chooses Right
Player 1 chooses Up	4, 3	−1, −1
Player 1 chooses Down	0, 0	3, 4

Normal form or payoff matrix of a 2-player, 2-strategy game

The normal (or strategic form) game is usually represented by a matrix which shows the players, strategies, and payoffs (see the example to the right). More generally it can be represented by any function that associates a payoff for each player with every possible combination of actions. In the accompanying example there are two players; one chooses the row and the other chooses the column. Each player has two strategies, which are specified by the number of rows and the number of columns. The payoffs are provided in the interior. The first number is the payoff received by the row player (Player 1 in our example); the second is the payoff for the column player (Player 2 in our example). Suppose that Player 1 plays Up and that Player 2 plays Left. Then Player 1 gets a payoff of 4, and Player 2 gets 3.

When a game is presented in normal form, it is presumed that each player acts simultaneously or, at least, without knowing the actions of the other. If players have some information about the choices of other players, the game is usually presented in extensive form.

Every extensive-form game has an equivalent normal-form game, however, the transformation to normal form may result in an exponential blowup in the size of the representation, making it computationally impractical.[48]

Characteristic function form[edit]

Main article: Cooperative game theory

In games that possess removable utility, separate rewards are not given; rather, the characteristic function decides the payoff of each unity. The idea is that the unity that is 'empty', so to speak, does not receive a reward at all.

The origin of this form is to be found in John von Neumann and Oskar Morgenstern's book; when looking at these instances, they guessed that when a union C {\displaystyle \mathbf {C} } appears, it works against the fraction (N C) {\displaystyle \left({\frac {\mathbf {N} }{\mathbf {C} }}\right)} as if two individuals were playing a normal game. The balanced payoff of C is a basic function. Although there are differing examples that help determine coalitional amounts from normal games, not all appear that in their function form can be derived from such.

Formally, a characteristic function is seen as: (N,v), where N represents the group of people and v : 2 N → R {\displaystyle v:2^{N}\to \mathbf {R} } is a normal utility.

Such characteristic functions have expanded to describe games where there is no removable utility.

Alternative game representations[edit]

See also: Succinct game

Alternative game representation forms are used for some subclasses of games or adjusted to the needs of interdisciplinary research.[49] In addition to classical game representations, some of the alternative representations also encode time related aspects.

Name	Year	Means	Type of games	Time
Congestion game[50]	1973	functions	subset of n-person games, simultaneous moves	No
Sequential form[51]	1994	matrices	2-person games of imperfect information	No

Name	Year	Means	Type of games	Time
Timed games[52][53]	1994	functions	2-person games	Yes
Gala[54]	1997	logic	n-person games of imperfect information	No
Local effect games[55]	2003	functions	subset of n-person games, simultaneous moves	No
GDL[56]	2005	logic	deterministic n-person games, simultaneous moves	No
Game Petri-nets[57]	2006	Petri net	deterministic n-person games, simultaneous moves	No
Continuous games[58]	2007	functions	subset of 2-person games of imperfect information	Yes
PNSI[59][60]	2008	Petri net	n-person games of imperfect information	Yes
Action graph games[61]	2012	graphs, functions	n-person games, simultaneous moves	No
Graphical games[62]	2015	graphs, functions	n-person games, simultaneous moves	No

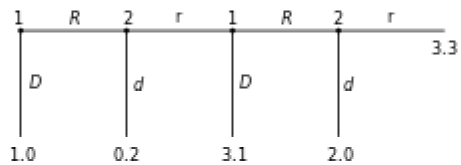
General and applied uses[edit]

As a method of applied mathematics, game theory has been used to study a wide variety of human and animal behaviors. It was initially developed in economics to understand a large collection of economic behaviors, including behaviors of firms, markets, and consumers. The first use of game-theoretic analysis was by Antoine Augustin Cournot in 1838 with his solution of the Cournot duopoly. The use of game theory in the social sciences has expanded, and game theory has been applied to political, sociological, and psychological behaviors as well.[63]

Although pre-twentieth-century naturalists such as Charles Darwin made game-theoretic kinds of statements, the use of game-theoretic analysis in biology began with Ronald Fisher's studies of animal behavior during the 1930s. This work predates the name "game theory", but it shares many important features with this field. The developments in economics were later applied to biology largely by John Maynard Smith in his 1982 book Evolution and the Theory of Games.[64]

In addition to being used to describe, predict, and explain behavior, game theory has also been used to develop theories of ethical or normative behavior and to prescribe such behavior.[65] In economics and philosophy, scholars have applied game theory to help in the understanding of good or proper behavior. Game-theoretic arguments of this type can be found as far back as Plato.[66] An alternative version of game theory, called chemical game theory, represents the player's choices as metaphorical chemical reactant molecules called "knowlecules".[67] Chemical game theory then calculates the outcomes as equilibrium solutions to a system of chemical reactions.

Description and modeling[edit]



A four-stage centipede game

The primary use of game theory is to describe and model how human populations behave.[citation needed] Some[who?] scholars believe that by finding the equilibria of games they can predict how actual human populations will behave when confronted with situations analogous to the game being studied. This particular view of game theory has been criticized. It is argued that the assumptions made by game theorists are often violated when applied to real-world situations. Game theorists usually assume players act rationally, but in practice, human behavior often deviates from this model. Game theorists respond by comparing their assumptions to those used in physics. Thus while their assumptions do not always hold, they can treat game theory as a reasonable scientific ideal akin to the models used by physicists. However, empirical work has shown that in some classic games, such as the centipede game, guess 2/3 of the average game, and the dictator game, people regularly do not play Nash equilibria. There is an ongoing debate regarding the importance of these experiments and whether the analysis of the experiments fully captures all aspects of the relevant situation.[b]

Some game theorists, following the work of John Maynard Smith and George R. Price, have turned to evolutionary game theory in order to resolve these issues. These models presume either no rationality or bounded rationality on the part of players. Despite the name, evolutionary game theory does not necessarily presume natural selection in the biological sense. Evolutionary game theory includes both biological as well as cultural evolution and also models of individual learning (for example, fictitious play dynamics).

Prescriptive or normative analysis[edit]

	Cooperate	Defect
Cooperate	-1, -1	-10, 0
Defect	0, -10	-5, -5

The prisoner's dilemma

Some scholars see game theory not as a predictive tool for the behavior of human beings, but as a suggestion for how people ought to behave. Since a strategy, corresponding to a Nash equilibrium of a game constitutes one's best response to the actions of the other players – provided they are in (the same) Nash equilibrium – playing a strategy that is part of a Nash equilibrium seems appropriate. This normative use of game theory has also come under criticism.[citation needed]

Economics and business[edit]

Game theory is a major method used in mathematical economics and business for modeling competing behaviors of interacting agents.[c][69][70][71] Applications include a wide array of economic phenomena and approaches, such as auctions, bargaining, mergers and acquisitions pricing,[72] fair division, duopolies, oligopolies, social network formation, agent-based computational economics,[73][74] general equilibrium, mechanism design,[75][76][77][78][79] and voting systems;[80] and across such broad areas as experimental economics,[81][82][83][84][85] behavioral economics,[86][87][88][89][90][91] information economics,[42][43][44][45] industrial organization,[92][93][94][95] and political economy.[96][97][98][99]

This research usually focuses on particular sets of strategies known as "solution concepts" or "equilibria". A common assumption is that players act rationally. In non-cooperative games, the most famous of these is the Nash equilibrium. A set of strategies is a Nash equilibrium if each represents a best response to the other strategies. If all the players are playing the strategies in

a Nash equilibrium, they have no unilateral incentive to deviate, since their strategy is the best they can do given what others are doing.[100][101]
The payoffs of the game are generally taken to represent the utility of individual players.
A prototypical paper on game theory in economics begins by presenting a game that is an abstraction of a particular economic situation. One or more solution concepts are chosen, and the author demonstrates which strategy sets in the presented game are equilibria of the appropriate type. Economists and business professors suggest two primary uses (noted above): descriptive and prescriptive.[65]
The Chartered Institute of Procurement & Supply (CIPS) promotes knowledge and use of game theory within the context of business procurement.[102] CIPS and TWS Partners have conducted a series of surveys designed to explore the understanding, awareness and application of game theory among procurement professionals. Some of the main findings in their third annual survey (2019) include:
application of game theory to procurement activity has increased – at the time it was at 19% across all survey respondents
65% of participants predict that use of game theory applications will grow
70% of respondents say that they have "only a basic or a below basic understanding" of game theory
20% of participants had undertaken on-the-job training in game theory
50% of respondents said that new or improved software solutions were desirable
90% of respondents said that they do not have the software they need for their work.[103]
Project management[edit]
Sensible decision-making is critical for the success of projects. In project management, game theory is used to model the decision-making process of players, such as investors, project managers, contractors, sub-contractors, governments and customers. Quite often, these players have competing interests, and sometimes their interests are directly detrimental to other players, making project management scenarios well-suited to be modeled by game theory.

Piraveenan (2019)[104] in his review provides several examples where game theory is used to model project management scenarios. For instance, an investor typically has several investment options, and each option will likely result in a different project, and thus one of the investment options has to be chosen before the project charter can be produced. Similarly, any large project involving subcontractors, for instance, a construction project, has a complex interplay between the main contractor (the project manager) and subcontractors, or among the subcontractors themselves, which typically has several decision points. For example, if there is an ambiguity in the contract between the contractor and subcontractor, each must decide how hard to push their case without jeopardizing the whole project, and thus their own stake in it. Similarly, when projects from competing organizations are launched, the marketing personnel have to decide what is the best timing and strategy to market the project, or its resultant product or service, so that it can gain maximum traction in the face of competition. In each of these scenarios, the required decisions depend on the decisions of other players who, in some way, have competing interests to the interests of the decision-maker, and thus can ideally be modeled using game theory.
Piraveenan[104] summarises that two-player games are predominantly used to model project management scenarios, and based on the identity of these players, five distinct types of games are used in project management.
Government-sector–private-sector games (games that model public–private partnerships)
Contractor–contractor games
Contractor–subcontractor games
Subcontractor–subcontractor games
Games involving other players
In terms of types of games, both cooperative as well as non-cooperative, normal-form as well as extensive-form, and zero-sum as well as non-zero-sum are used to model various project management scenarios.
Political science[edit]
The application of game theory to political science is focused in the overlapping areas of fair division, political economy, public choice, war bargaining, positive political theory, and social choice theory. In each of these

areas, researchers have developed game-theoretic models in which the players are often voters, states, special interest groups, and politicians.
Early examples of game theory applied to political science are provided by Anthony Downs. In his 1957 book An Economic Theory of Democracy,[105] he applies the Hotelling firm location model to the political process. In the Downsian model, political candidates commit to ideologies on a one-dimensional policy space. Downs first shows how the political candidates will converge to the ideology preferred by the median voter if voters are fully informed, but then argues that voters choose to remain rationally ignorant which allows for candidate divergence. Game theory was applied in 1962 to the Cuban Missile Crisis during the presidency of John F. Kennedy.[106]
It has also been proposed that game theory explains the stability of any form of political government. Taking the simplest case of a monarchy, for example, the king, being only one person, does not and cannot maintain his authority by personally exercising physical control over all or even any significant number of his subjects. Sovereign control is instead explained by the recognition by each citizen that all other citizens expect each other to view the king (or other established government) as the person whose orders will be followed. Coordinating communication among citizens to replace the sovereign is effectively barred, since conspiracy to replace the sovereign is generally punishable as a crime. Thus, in a process that can be modeled by variants of the prisoner's dilemma, during periods of stability no citizen will find it rational to move to replace the sovereign, even if all the citizens know they would be better off if they were all to act collectively.[107]
A game-theoretic explanation for democratic peace is that public and open debate in democracies sends clear and reliable information regarding their intentions to other states. In contrast, it is difficult to know the intentions of nondemocratic leaders, what effect concessions will have, and if promises will be kept. Thus there will be mistrust and unwillingness to make concessions if at least one of the parties in a dispute is a non-democracy.[108]
However, game theory predicts that two countries may still go to war even if their leaders are cognizant of the costs of fighting. War may result from asymmetric information; two countries may have incentives to mis-represent the amount of military resources they have on hand, rendering them unable to settle disputes agreeably without resorting to fighting. Moreover, war may

arise because of commitment problems: if two countries wish to settle a dispute via peaceful means, but each wishes to go back on the terms of that settlement, they may have no choice but to resort to warfare. Finally, war may result from issue indivisibilities.[109]

Game theory could also help predict a nation's responses when there is a new rule or law to be applied to that nation. One example is Peter John Wood's (2013) research looking into what nations could do to help reduce climate change. Wood thought this could be accomplished by making treaties with other nations to reduce greenhouse gas emissions. However, he concluded that this idea could not work because it would create a prisoner's dilemma for the nations.[110]

Biology[edit]

Main article: Evolutionary game theory

	Hawk	Dove
Hawk	20, 20	80, 40
Dove	40, 80	60, 60

The hawk-dove game

Unlike those in economics, the payoffs for games in biology are often interpreted as corresponding to fitness. In addition, the focus has been less on equilibria that correspond to a notion of rationality and more on ones that would be maintained by evolutionary forces. The best-known equilibrium in biology is known as the evolutionarily stable strategy (ESS), first introduced in (Maynard Smith & Price 1973). Although its initial motivation did not involve any of the mental requirements of the Nash equilibrium, every ESS is a Nash equilibrium.

In biology, game theory has been used as a model to understand many different phenomena. It was first used to explain the evolution (and stability) of the approximate 1:1 sex ratios. (Fisher 1930) harv error: no target: CITEREFFisher1930 (help) suggested that the 1:1 sex ratios are a result of evolutionary forces acting on individuals who could be seen as trying to maximize their number of grandchildren.

Additionally, biologists have used evolutionary game theory and the ESS to explain the emergence of animal communication.[111] The analysis of signaling games and other communication games has provided insight into the evolution of communication among animals. For example, the mobbing behavior of many species, in which a large number of prey animals attack a larger predator, seems to be an example of spontaneous emergent

organization. Ants have also been shown to exhibit feed-forward behavior akin to fashion (see Paul Ormerod's Butterfly Economics).
Biologists have used the game of chicken to analyze fighting behavior and territoriality.[112]
According to Maynard Smith, in the preface to Evolution and the Theory of Games, "paradoxically, it has turned out that game theory is more readily applied to biology than to the field of economic behaviour for which it was originally designed". Evolutionary game theory has been used to explain many seemingly incongruous phenomena in nature.[113]
One such phenomenon is known as biological altruism. This is a situation in which an organism appears to act in a way that benefits other organisms and is detrimental to itself. This is distinct from traditional notions of altruism because such actions are not conscious, but appear to be evolutionary adaptations to increase overall fitness. Examples can be found in species ranging from vampire bats that regurgitate blood they have obtained from a night's hunting and give it to group members who have failed to feed, to worker bees that care for the queen bee for their entire lives and never mate, to vervet monkeys that warn group members of a predator's approach, even when it endangers that individual's chance of survival.[114] All of these actions increase the overall fitness of a group, but occur at a cost to the individual.
Evolutionary game theory explains this altruism with the idea of kin selection. Altruists discriminate between the individuals they help and favor relatives. Hamilton's rule explains the evolutionary rationale behind this selection with the equation $c < b \times r$, where the cost c to the altruist must be less than the benefit b to the recipient multiplied by the coefficient of relatedness r. The more closely related two organisms are causes the incidences of altruism to increase because they share many of the same alleles. This means that the altruistic individual, by ensuring that the alleles of its close relative are passed on through survival of its offspring, can forgo the option of having offspring itself because the same number of alleles are passed on. For example, helping a sibling (in diploid animals) has a coefficient of $1/2$, because (on average) an individual shares half of the alleles in its sibling's offspring. Ensuring that enough of a sibling's offspring survive to adulthood precludes the necessity of the altruistic individual producing offspring.[114] The coefficient values

depend heavily on the scope of the playing field; for example if the choice of whom to favor includes all genetic living things, not just all relatives, we assume the discrepancy between all humans only accounts for approximately 1% of the diversity in the playing field, a coefficient that was 1⁄2 in the smaller field becomes 0.995. Similarly if it is considered that information other than that of a genetic nature (e.g. epigenetics, religion, science, etc.) persisted through time the playing field becomes larger still, and the discrepancies smaller.

Computer science and logic[edit]

Game theory has come to play an increasingly important role in logic and in computer science. Several logical theories have a basis in game semantics. In addition, computer scientists have used games to model interactive computations. Also, game theory provides a theoretical basis to the field of multi-agent systems.[115]

Separately, game theory has played a role in online algorithms; in particular, the k-server problem, which has in the past been referred to as games with moving costs and request-answer games.[116] Yao's principle is a game-theoretic technique for proving lower bounds on the computational complexity of randomized algorithms, especially online algorithms.

The emergence of the Internet has motivated the development of algorithms for finding equilibria in games, markets, computational auctions, peer-to-peer systems, and security and information markets. Algorithmic game theory[117] and within it algorithmic mechanism design[118] combine computational algorithm design and analysis of complex systems with economic theory.[119][120][121]

Philosophy[edit]

	Stag	Hare
Stag	3, 3	0, 2
Hare	2, 0	2, 2

Stag hunt

Game theory has been put to several uses in philosophy. Responding to two papers by W.V.O. Quine (1960, 1967), Lewis (1969) used game theory to develop a philosophical account of convention. In so doing, he provided the first analysis of common knowledge and employed it in analyzing play in coordination games. In addition, he first suggested that one can understand meaning in terms of signaling games. This later suggestion has been pursued by several philosophers since

Lewis.[122][123] Following Lewis (1969) game-theoretic account of conventions, Edna Ullmann-Margalit (1977) and Bicchieri (2006) have developed theories of social norms that define them as Nash equilibria that result from transforming a mixed-motive game into a coordination game.[124][125]

Game theory has also challenged philosophers to think in terms of interactive epistemology: what it means for a collective to have common beliefs or knowledge, and what are the consequences of this knowledge for the social outcomes resulting from the interactions of agents. Philosophers who have worked in this area include Bicchieri (1989, 1993),[126][127] Skyrms (1990),[128] and Stalnaker (1999).[129]

In ethics, some (most notably David Gauthier, Gregory Kavka, and Jean Hampton)[who?] authors have attempted to pursue Thomas Hobbes' project of deriving morality from self-interest. Since games like the prisoner's dilemma present an apparent conflict between morality and self-interest, explaining why cooperation is required by self-interest is an important component of this project. This general strategy is a component of the general social contract view in political philosophy (for examples, see Gauthier (1986) and Kavka (1986) harvtxt error: no target: CITEREFKavka1986 (help)).[d]

Other authors have attempted to use evolutionary game theory in order to explain the emergence of human attitudes about morality and corresponding animal behaviors. These authors look at several games including the prisoner's dilemma, stag hunt, and the Nash bargaining game as providing an explanation for the emergence of attitudes about morality (see, e.g., Skyrms (1996, 2004) and Sober and Wilson (1998)).

Retail and consumer product pricing[edit]

Game theory applications are often used in the pricing strategies of retail and consumer markets, particularly for the sale of inelastic goods. With retailers constantly competing against one another for consumer market share, it has become a fairly common practice for retailers to discount certain goods, intermittently, in the hopes of increasing foot-traffic in brick and mortar locations (websites visits for e-commerce retailers) or increasing sales of ancillary or complimentary products.[130]

Black Friday, a popular shopping holiday in the US, is when many retailers focus on optimal pricing strategies to capture the holiday shopping market. In the Black Friday scenario, retailers using game theory applications typically ask "what is the dominant competitor's reaction to me?"[131] In such a scenario, the game has two players: the retailer, and the consumer. The retailer is focused on an optimal pricing strategy, while the consumer is focused on the best deal. In this closed system, there often is no dominant strategy as both players have alternative options. That is, retailers can find a different customer, and consumers can shop at a different retailer.[131] Given the market competition that day, however, the dominant strategy for retailers lies in outperforming competitors. The open system assumes multiple retailers selling similar goods, and a finite number of consumers demanding the goods at an optimal price. A blog by a Cornell University professor provided an example of such a strategy, when Amazon priced a Samsung TV $100 below retail value, effectively undercutting competitors. Amazon made up part of the difference by increasing the price of HDMI cables, as it has been found that consumers are less price discriminatory when it comes to the sale of secondary items.[131]

Retail markets continue to evolve strategies and applications of game theory when it comes to pricing consumer goods. The key insights found between simulations in a controlled environment and real-world retail experiences show that the applications of such strategies are more complex, as each retailer has to find an optimal balance between pricing, supplier relations, brand image, and the potential to cannibalize the sale of more profitable items.[132]

Epidemiology[edit]

Since the decision to take a vaccine for a particular disease is often made by individuals, who may consider a range of factors and parameters in making this decision (such as the incidence and prevalence of the disease, perceived and real risks associated with contracting the disease, mortality rate, perceived and real risks associated with vaccination, and financial cost of vaccination), game theory has been used to model and predict vaccination uptake in a society.[133][134]

In popular culture[edit]

Based on the 1998 book by Sylvia Nasar,[135] the life story of game theorist and mathematician John Nash was turned into the 2001 biopic A Beautiful Mind, starring Russell Crowe as Nash.[136]

The 1959 military science fiction novel Starship Troopers by Robert A. Heinlein mentioned "games theory" and "theory of games".[137] In the 1997 film of the same name, the character Carl Jenkins referred to his military intelligence assignment as being assigned to "games and theory".

The 1964 film Dr. Strangelove satirizes game theoretic ideas about deterrence theory. For example, nuclear deterrence depends on the threat to retaliate catastrophically if a nuclear attack is detected. A game theorist might argue that such threats can fail to be credible, in the sense that they can lead to subgame imperfect equilibria. The movie takes this idea one step further, with the Soviet Union irrevocably committing to a catastrophic nuclear response without making the threat public.[138]

The 1980s power pop band Game Theory was founded by singer/songwriter Scott Miller, who described the band's name as alluding to "the study of calculating the most appropriate action given an adversary ... to give yourself the minimum amount of failure".[139]

Liar Game, a 2005 Japanese manga and 2007 television series, presents the main characters in each episode with a game or problem that is typically drawn from game theory, as demonstrated by the strategies applied by the characters.[citation needed]

The 1974 novel Spy Story by Len Deighton explores elements of Game Theory in regard to cold war army exercises.

The 2008 novel The Dark Forest by Liu Cixin explores the relationship between extraterrestrial life, humanity, and game theory.

The prime antagonist Joker in the movie The Dark Knight presents game theory concepts—notably the prisoner's dilemma in a scene where he asks passengers in two different ferries to bomb the other one to save their own.